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Duopoly — Part II: Lost in Fractals

Summary:

In his fascinating book Puu (2000) shows that the dynamics of a very simple Cournot duopoly (with constant marginal costs and an isoelastic demand curve) may lead to an instable Nash-Cournot equilibrium. Furthermore a lot of "exotic" phenomena may appear like Hopf bifurcation, saddle-node bifurcation and fractal attractors.

1. A static Cournot duopoly

There are two competitors producing the same homogenous good. Their supply is denoted as x and y. An **isoelastic (invers) demand function** is assumed:

$$p(x,y) := \frac{1}{x+y}$$

The duopolists produce with **constant marginal costs** $a, b>0$. Ignoring fixed costs the **profit functions** become:

$$\Pi_x(x,y,a) := \frac{x}{x+y} - a\cdot x$$

$$\Pi_y(x,y,b) := \frac{y}{x+y} - b\cdot y$$

Equating the partial derivatives to 0 ...

$$\frac{d}{dx}\Pi_x(x,y,a) = 0 \;\Bigg|\; \begin{array}{l}\text{auflösen}, x\\ \text{vereinfachen}\end{array} \rightarrow \begin{bmatrix} \dfrac{-\left[a\cdot y - (a\cdot y)^{\left(\frac{1}{2}\right)}\right]}{a} \\ \dfrac{-\left[a\cdot y + (a\cdot y)^{\left(\frac{1}{2}\right)}\right]}{a} \end{bmatrix}$$

$$\frac{d}{dy}\Pi_y(x,y,b) = 0 \;\Bigg|\; \begin{array}{l}\text{auflösen}, y\\ \text{vereinfachen}\end{array} \rightarrow \begin{bmatrix} \dfrac{-\left[b\cdot x - (b\cdot x)^{\left(\frac{1}{2}\right)}\right]}{b} \\ \dfrac{-\left[b\cdot x + (b\cdot x)^{\left(\frac{1}{2}\right)}\right]}{b} \end{bmatrix}$$

... we can solve for the **reaction functions** provided that the quantities are positive ...

$$R_x(y,a) := \sqrt{\frac{y}{a}} - y$$

$$R_y(x,b) := \sqrt{\frac{x}{b}} - x$$

... and obtain the **Cournot-Nash equilibrium** (x_C, y_C) from:

$$\left(R_x\left(\sqrt{\frac{x}{b}} - x, a\right) = x\right) \;\Big|\; \begin{array}{l}\text{auflösen}, x\\ \text{vereinfachen}\end{array} \rightarrow \begin{bmatrix} 0 \\ \dfrac{1}{(b+a)^2}\cdot b \end{bmatrix}$$

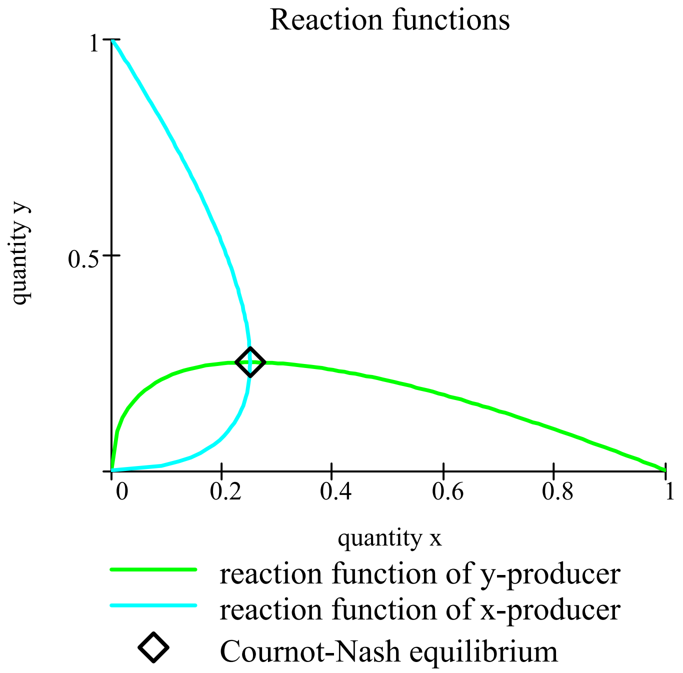
$$\Rightarrow \qquad x_C(a,b) := \frac{b}{(a+b)^2}$$

$$y_C(a,b) := \frac{a}{(a+b)^2}$$

Numerical example

Enter marginal costs: $a := 1$ $b := 1$

Adjust range of plot: $x_{max} := 1$ $y_{max} := 1$



Cournot-Nash equilibrium:

$x_C(a,b) = 0.25$

$y_C(a,b) = 0.25$

2. Iterative adjustment

Now assume a **lagged reaction** of the duopolists with

$$x_i = \sqrt{\frac{y_{i-1}}{a}} - y_{i-1} \qquad \text{and}$$

$$y_i = \sqrt{\frac{x_{i-1}}{b}} - x_{i-1}$$

The fixed point of this iterative process is the Cournot-Nash equilibrium. **Stability** of this equilibrium is ensured when (Puu 2000, 245 - 246):

$$3 - \left(\sqrt{2}\right)\cdot 2 < \left(\frac{a}{b}, \frac{b}{a}\right) < 3 + \left(\sqrt{2}\right)\cdot 2$$

Therefore, the ratio of the marginal costs marks the critical parameter of this process. For further considerations let $b := 1$. Thus only the marginal cost "a" becomes the critical parameter.

$$\text{stability\_check}(a) := \begin{cases} \text{"stable Cournot point!"} & \text{if } \left[3 - \left(\sqrt{2}\right)\cdot 2 < a < 3 + \left(\sqrt{2}\right)\cdot 2\right] = 1 \\ \text{"unstable Cournot point"} & \text{otherwise} \end{cases}$$

Now an example of chaotic production cycles is given. You may change the marginal cost parameter (where $0.16 \le a \le 6.25$) to get stable cycles around or convergence to the Cournot equilibrium. (Try for example a = 1, 0.18, 0.162, 0.161, 6, 6.25).

Enter the **marginal cost parameter:**

$a := .16$ $\Rightarrow$ stability_check(a) = "unstable Cournot point"

Actually, it has no importance at all whether both firms adjust simultaneously or take turns in their adjustments. The only difference is how the (essentially autonomous) time series of x and y are paired together. To start with an iteration, let us assume that in the beginning the x-producer supplies

$$x_0 := .005$$

and simultaneously the y-producer responses with

$$y_0 := \left(\sqrt{x_0} - x_0\right)$$

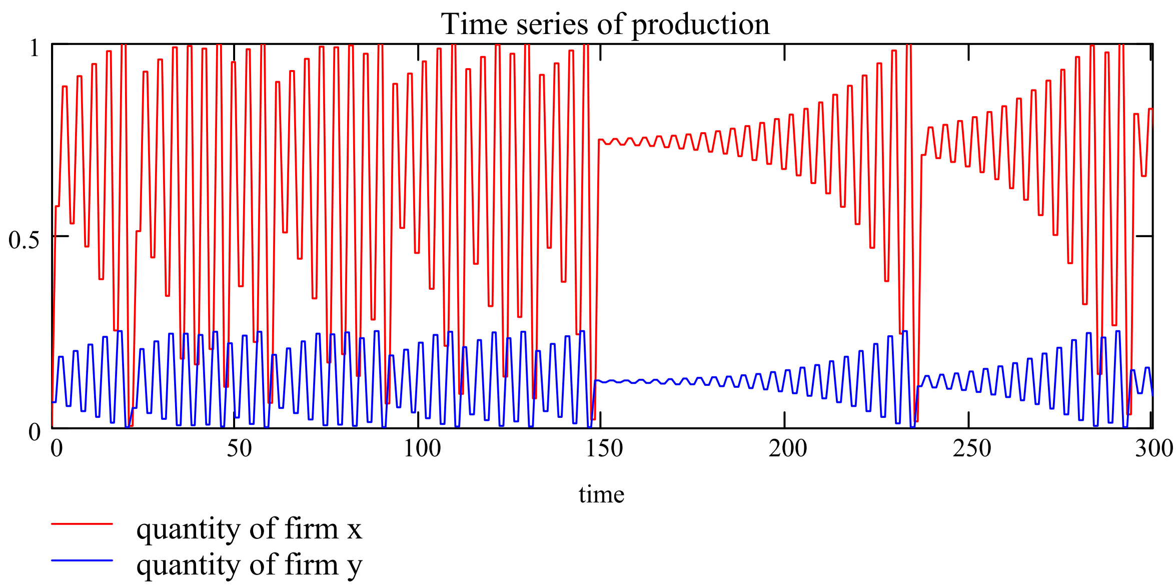
Given the maximum number of iterations $T_{max} := 1000$ the iterative adjustment follows

$i := 1 .. T_{max}$

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} := \begin{bmatrix} \sqrt{\frac{y_{i-1}}{a}} - y_{i-1} \\ \sqrt{x_{i-1}} - x_{i-1} \end{bmatrix}$$

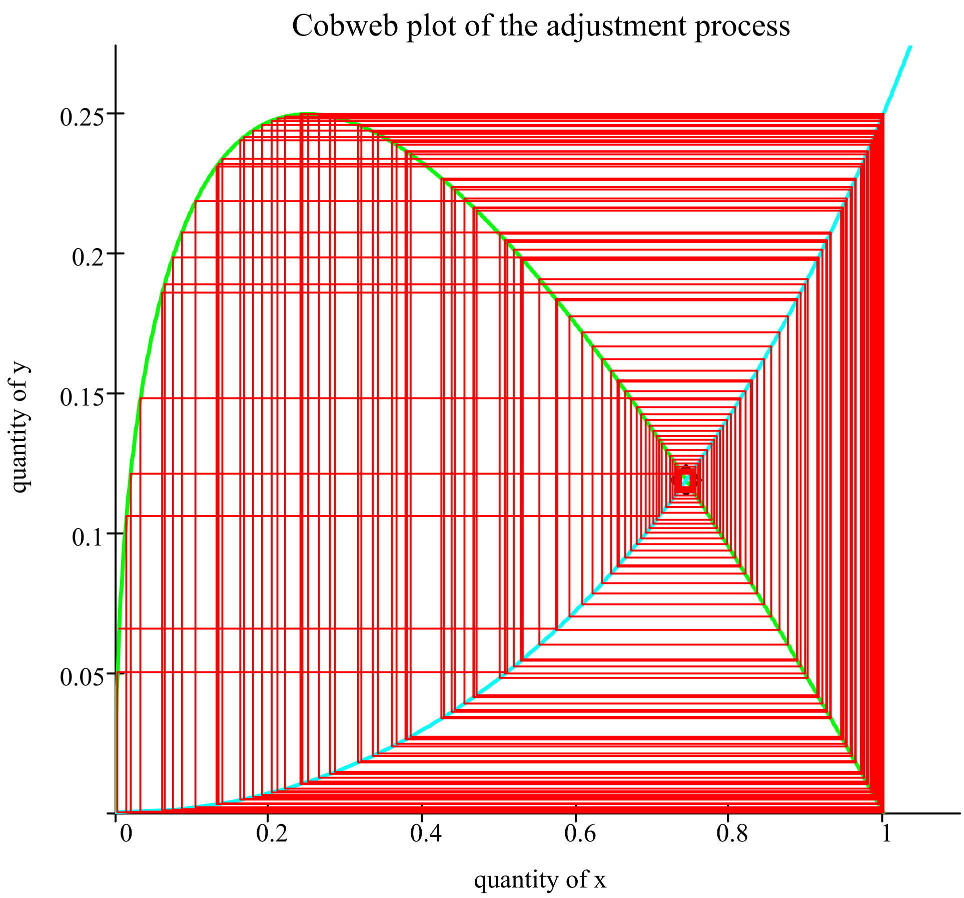
Range of plotted time periods: $t_{begin} := 0$ $t_{stop} := 300$

$i := 0 .. T_{max}$



The adjustment process can be described also by a cobweb diagram, where the stepwise decisions of the duopolists are drawn as lines connecting both reaction curves.

Range of plotted time periods: $t_{begin} := 0$ $t_{stop} := 300$



Because we are dealing with a pair of independent iterations, we then get iterations of each of the variables alone, without interference of the other one (though the lag is now two periods!). For variable x this results in:

$$x_i = \sqrt{\frac{\sqrt{x_{i-2}} - x_{i-2}}{a}} - \left(\sqrt{x_{i-2}} - x_{i-2}\right)$$

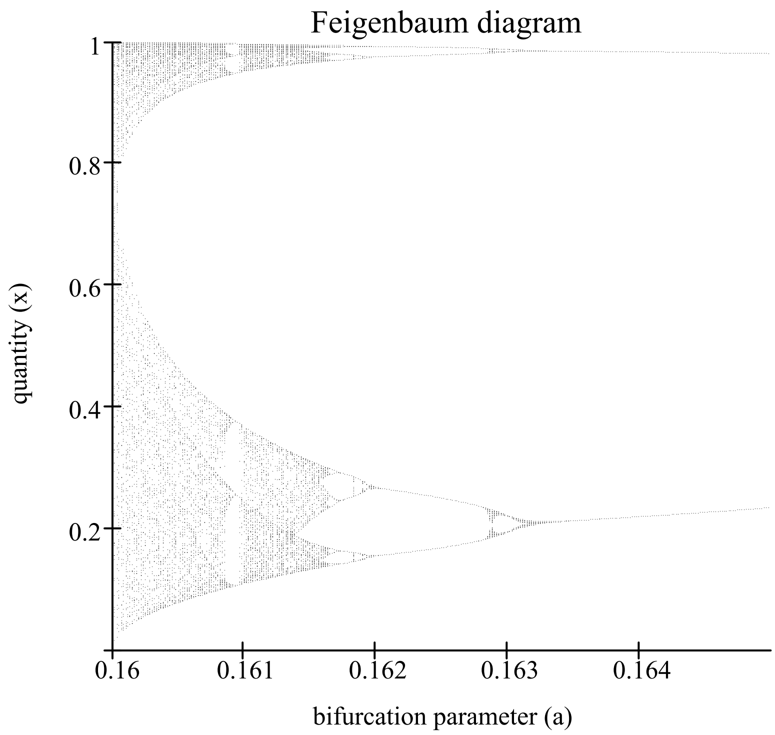
This single difference equation of 2nd order reproduces the characteristics of the dynamics of the 2-variable system. Therefore, bifurcations can be observed by plotting the time path of the variable x against different values of the critical parameter a into a **Feigenbaum diagram**.

Resolution of graph: $RES := 3$ $(1, 2, .., 10)$

Range of plotted values: $a_{bottom} := .16$ $a_{top} := .165$

$x_{bottom} := 0$ $x_{top} := 1$

(Try another interval of the parameter a, for example [6.1, 6.25] *with* $x_{top} = .05$*.)*



3. Adaptive expectations

Assume now that both firms do not immediately reach their new optimal positions, but adjust their previous decisions in the direction of the new optimum with the **adjustment speeds** λ and μ:

$$x_i = x_{i-1} + \lambda \cdot \left(\sqrt{\frac{y_{i-1}}{a}} - y_{i-1} - x_{i-1} \right) \qquad \text{and}$$

$$y_i = y_{i-1} + \mu \cdot \left(\sqrt{\frac{x_{i-1}}{b}} - x_{i-1} - y_{i-1} \right) \qquad \text{with} \qquad 0 \le \lambda , \mu \le 1$$

The **stability** of the Cournot fixed point is checked by (Puu 2000, 249 - 250):

$$\text{stability\_check}(a,b,\lambda,\mu) := \begin{cases} \text{"stable Cournot point!"} & \text{if } (a-b)^2 < 4 \cdot a \cdot b \cdot \left(\frac{1}{\lambda} + \frac{1}{\mu} - 1 \right) \\ \text{"unstable Cournot point!"} & \text{otherwise} \end{cases}$$

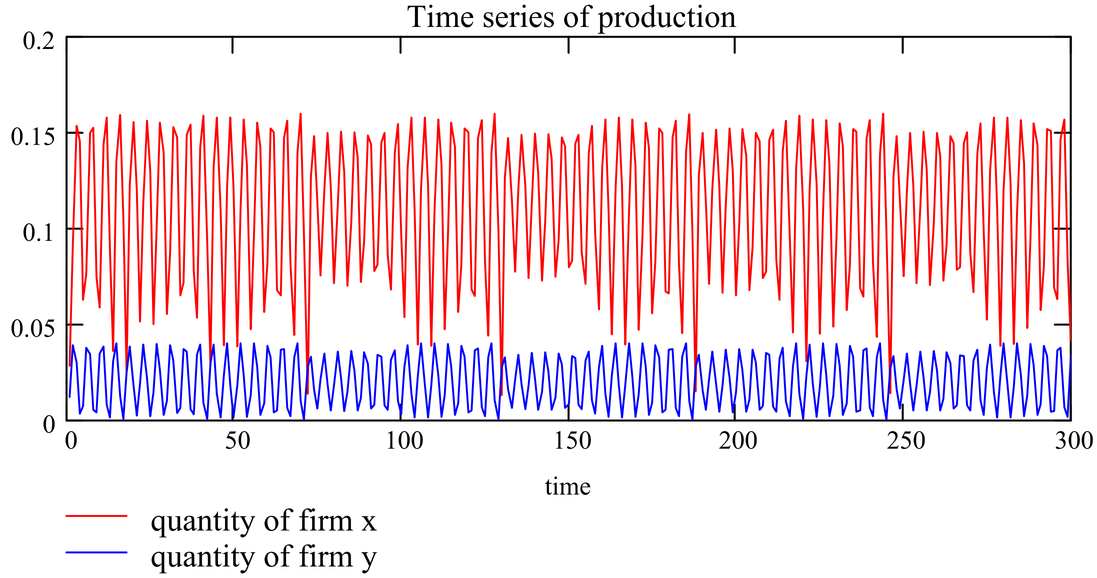
Initial values: $x_0 := .001$ $\qquad y_0 := .001$

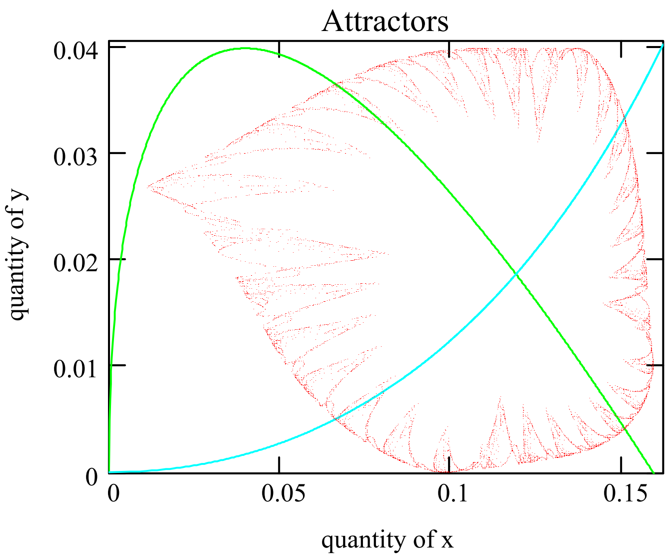
Maximum number of time periods: $T_{max} := 20000$

$i := 1 .. T_{max}$

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} := \begin{bmatrix} x_{i-1} + \lambda \cdot \left(\sqrt{\frac{y_{i-1}}{a}} - y_{i-1} - x_{i-1} \right) \\ y_{i-1} + \mu \cdot \left(\sqrt{\frac{x_{i-1}}{b}} - x_{i-1} - y_{i-1} \right) \end{bmatrix}$$

Range of plotted time periods: $t_{begin} := 0$ $t_{stop} := 300$





Enter parameters:

$\lambda \equiv .89899$

$\mu \equiv 1.0$

$b_{new} \equiv 6.2812$

$a_{new} \equiv .98101$

stability_check(a, b, λ, μ) = "unstable Cournot point!"

With the pre adjusted parameters from above you will get a fractal attractor in the nice shape of a "leaf". To find a Hopf bifurcation, saddle-node bifurcations and another strange attractor use $b_{new} = \mu = 1$, $a_{new} = 0.16$ and vary λ from 0.9 to 1 in small steps.

Literature:

- ***Puu, T.:*** *Attractors, Bifurcations, and Chaos. Nonlinear Phenomena in Economics. Berlin et al. 2000.*