FH-Kiel
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[redacted]

Markov Chains

Summary:

This worksheet offers some simple tools to handle with Markov chains. It will be shown how to compute the ergodic distribution and to generate random simulations.

Introduction (with Example 1)

A **stochastic process** is a sequence of random vectors. If we study discrete time models this sequence can be ordered by a time index k, taken to be integers in this worksheet. A stochastic process $\{x_k\}$ is said to have the **Markov property** if for all $\tau \geq 1$ and all k

$$\text{Prob}(x_{k+1} \mid x_k, x_{k-1}, \ldots, x_{k-\tau}) = \text{Prob}(x_{k+1} \mid x_k)$$

Assuming this property we call such a sequence a **Markov chain** which is characterized by the following three objects.

1. There is a vector x which records the possible values of the **state of the system**; for example:

$$x := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2. There is a quadratic **transition matrix** P, which records the probabilities of moving from one value of the state to another in one period; for example:

$$P := \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.8 & 0.1 & 0.1 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}$$

3. There is a vector π_0 recording the probabilities (**initial distribution**) of being in each state at time k = 0; for example:

$$\pi_0 := \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Be sure that the single probabilities sum up to 1. You may check this using the following subroutine, which is helpful to control the input of a matrix P with many entries:

$$\text{validity}(P, \pi_0) := \left| \begin{array}{l} n \leftarrow \text{zeilen}(P) - 1 \\ \text{for } i \in 0..n \\ \quad \text{one}_i \leftarrow 1 \\ \text{"O.K." if } \left[\sum_{i=0}^{n} \left(\pi_0^{\,T}\right)_i = 1 \right] \cdot (P \cdot \text{one} = \text{one}) \\ \text{"These are no probability measures!" otherwise} \end{array} \right.$$

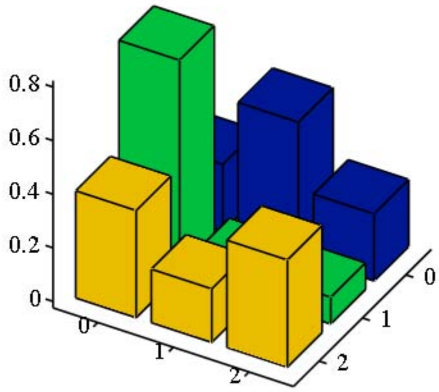
$$\text{validity}(P, \pi_0) = \text{"O.K."}$$

The probability of moving from state i to state j in k periods is $\left(P^k\right)_{i,j}$; try for example:

$k := 2$ $\quad i := 0$ $\quad j := 1$ $\quad \left(P^k\right)_{i,j} = 0.225$

Hence P^k is the **transition matrix for k periods**. What happens if you increase k step by step?

$k := 1$



$$P^k = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.8 & 0.1 & 0.1 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}$$

P^k

Increasing the exponent k, the matrix P^k converges very quickly, showing the same distributions in every row.

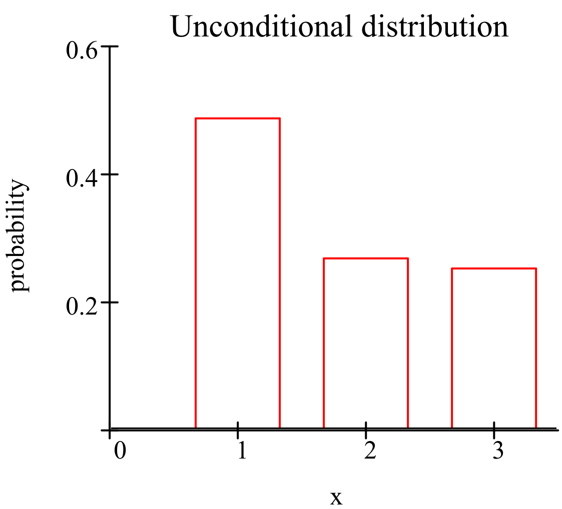
The **unconditional probability** distribution of x_i $(i := 0..2)$ after k periods is:

$k := 1$

$\pi_0 \cdot P^k = (0.4833333 \quad 0.2666667 \quad 0.25)$

with the (unconditional) **expectation:**

$\pi_0 \cdot P^k \cdot x = (1.7666667)$



Now rise k again. For high k this distribution equals the rows in P^k. This means that the initial distribution π_0 becomes meaningless if time passes by. Verify this for different initial distributions.

A distribution π is called **stationary** if it satisfies for all k

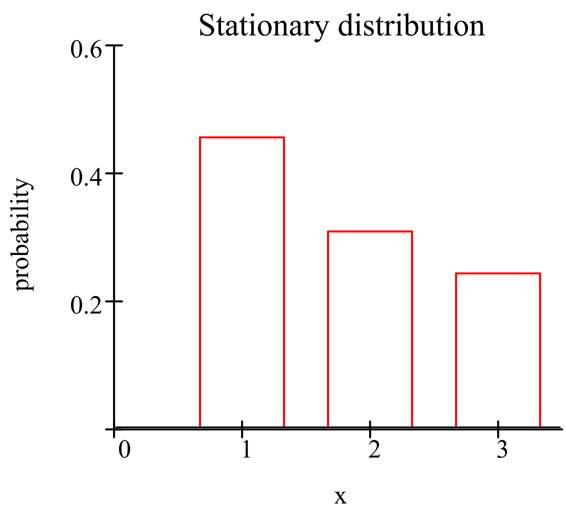
$$\text{Prob}\left(x_k\right) = \text{Prob}\left(x_{k-1}\right) = \pi$$

that is, if the distribution remains unaltered with the passage of time. Because the unconditional probability distributions evolve according to

$$\text{Prob}\left(x_k\right) = \text{Prob}\left(x_{k-1}\right) \cdot P$$

a stationary distribution must satisfy $\pi = \pi \cdot P$, which can be also expressed as the linear system $\pi \cdot (P - I) = 0$. However, this equation is homogenous linear and has no unique solution. But we know that π is fixed by the additional condition $\Sigma \pi_i = 1$. A small program helpes to solve the equation under this restriction:

```
π(P) := | n ← zeilen(P) − 1
        | Π ← P − einheit(n + 1)
        | for i ∈ 0 .. n
        |   Π[i,n ← 1
        | for i ∈ 0 .. n
        |   | ι[i ← 0  if i<n
        |   | ι[i ← 1  otherwise
        | "No unique solution!"  on error ι^T·Π^-1
```



$$\pi(P) = (0.4541485 \quad 0.3056769 \quad 0.2401747)$$

Compare this distribution with the unconditional distribution for many transition periods k.

If for all initial distributions π_0 it is true that $\lim_{k \to \infty} \pi_0 \cdot P^k$ converges all to the same π which satisfies $\pi \cdot (P - I) = 0$, we say that the Markov chain is **asymptotically stationary** with a unique invariant (i.e. ergodic) distribution.The following theorem can be used to show that a Markov chain is asymptotically stationary:

***Theorem*:** Let P a stochastic matrix with $\left(P_{i,j}\right)^k > 0$ for some value of k and all i and j. Then P has unique stationary distribution, and the process is asymptotically stationary.

To prepare the random **simulation** of the outcome from a Markov chain use these programs:

```
rdmultinom(π) := | π ← π^T
                 | n ← zeilen(π) − 1
                 | r ← rnd(1)
                 | p ← π_0
                 | z_0 ← 1  if 0 ≤ r < p
                 | z_0 ← 0  otherwise
                 | for i ∈ 1 .. n
                 |   | z_i ← 1  if p ≤ r < p + π_i
                 |   | z_i ← 0  otherwise
                 |   | p ← p + π_i
                 | z_n ← 1  if r = 1
                 | z
```

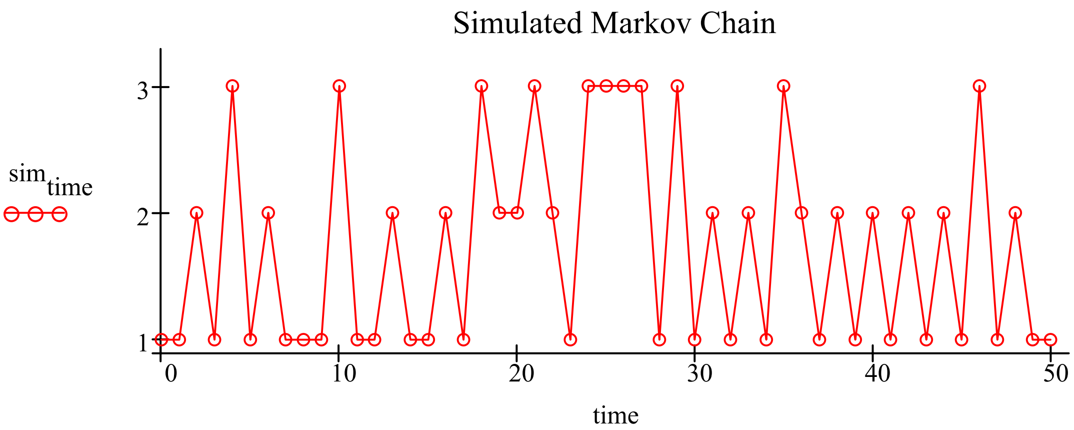
```
rdmarkov(k, P, π_0, x) := | n ← zeilen(P) − 1
                          | M^<0> ← rdmultinom(π_0)
                          | for τ ∈ 1 .. k
                          |   | i ← Σ_{j=0}^{n} j·(M_{j,τ−1} = 1)
                          |   | M^<τ> ← rdmultinom(((P^T)^<i>)^T)
                          | M^T·x
```

Now we are ready to start a simulation:

$k := 50$

$sim := rdmarkov(k, P, \pi_0, x)$ ⇐ Click with your mouse on the red field and press the F9-button to get another random selection!

$time := 0..k$



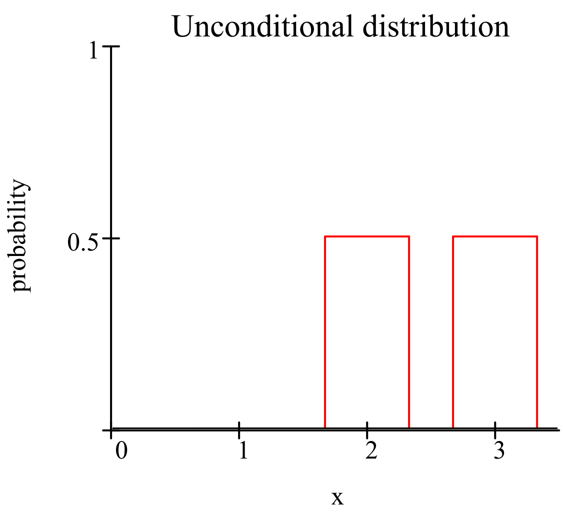
Example 2:

We call a state i a "**reflecting** state" if $P_{i,i} = 0$. In this example all states are reflecting:

$$P := \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{bmatrix} \qquad x := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \pi_0 := (1 \quad 0 \quad 0)$$

Now iterate the unconditional probabilities by increasing k.

$k := 1$



with the (unconditional) **expectation:**

$$\pi_0 \cdot P^k \cdot x = (2.5)$$

The probabilities jump alternately between state x_0 and x_2 converging to the stationary distribution:

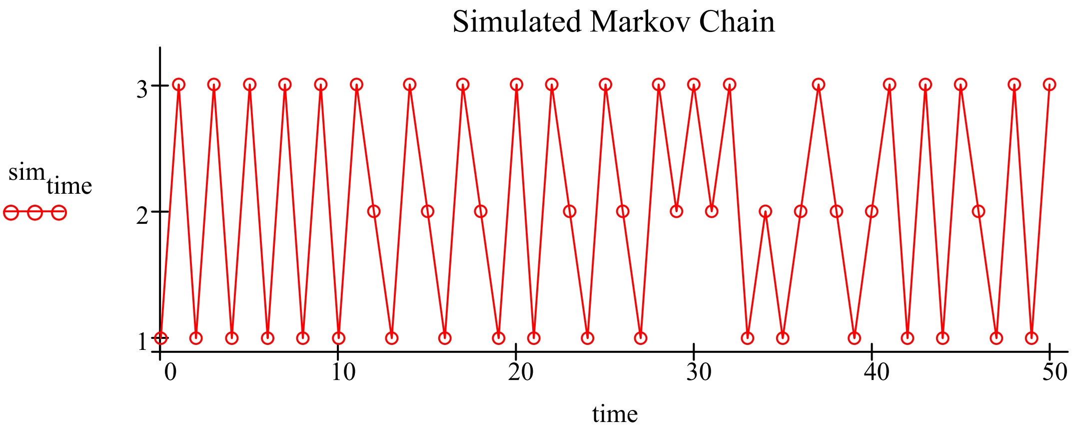
$\pi(P) = (0.3333333 \quad 0.3333333 \quad 0.3333333)$

A simulation of this process shows that one single state never appears in succession.

$k := 50$

$sim := rdmarkov(k, P, \pi_0, x)$ ⇐ Click with your mouse on the red field and press the F9-button to get another random selection!

$time := 0 .. k$



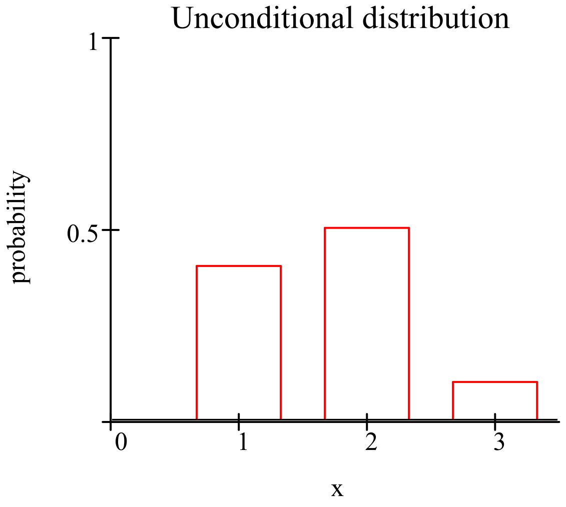
Example 3:

We call a state x_i an "**absorbing** state" if $P_{i,i} = 1$. In the following example this is state $x_2 = 3$:

$$P := \begin{bmatrix} .4 & .5 & .1 \\ .5 & .4 & .1 \\ 0 & 0 & 1 \end{bmatrix} \qquad x := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \pi_0 := (1 \quad 0 \quad 0)$$

Now iterate the unconditional probabilities by increasing k.

$k := 1$



with the (unconditional) **expectation:**

$\pi_0 \cdot P^k \cdot x = (1.7)$

After several steps this iteration converges to the stationary distribution:

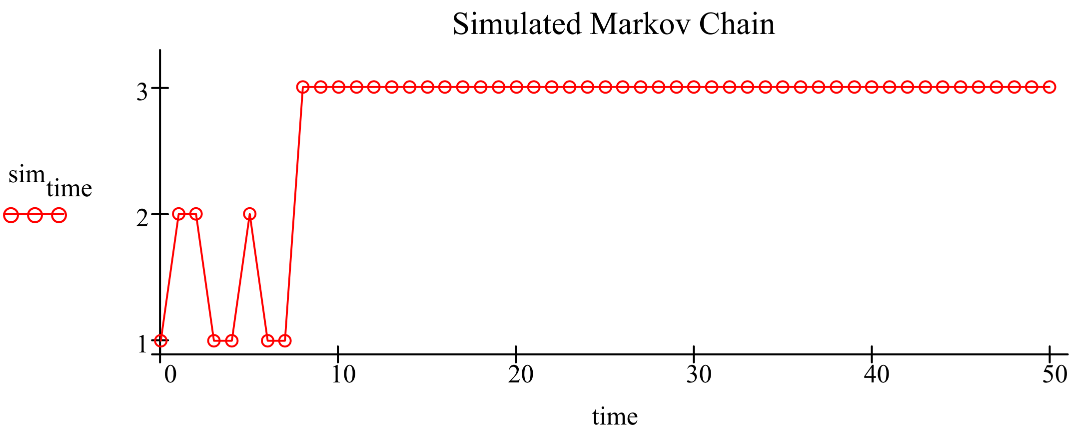
$\pi(P) = (0 \quad 0 \quad 1)$

That means that in the long run we end in state $x_2 = 3$. Simulations verify this result:

$k := 50$

$sim := rdmarkov(k, P, \pi_0, x)$ ⇐ Click with your mouse on the red field and press the F9-button to get another random selection!

$time := 0 .. k$



Example 4:

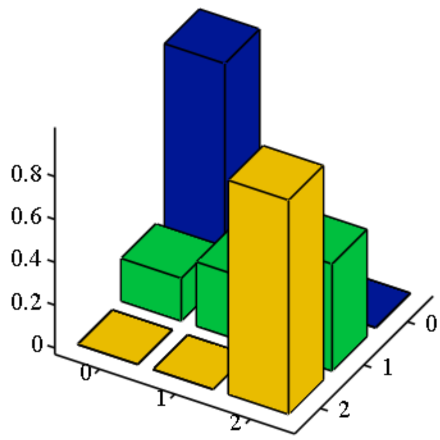
Let $P := \begin{bmatrix} 1 & 0 & 0 \\ .2 & .3 & .5 \\ 0 & 0 & 1 \end{bmatrix}$ $x := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\pi_0 := (1 \;\; 0 \;\; 0)$

There exists **no unique** stationary distribution:

$\pi(P) =$ "No unique solution!"

But there are 3 different stationary distributions. You will detect them by iterating P^k:

$k := 1$



P^k

$$P^k = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & 0.3 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 5:

Suppose that an individual earns m = 0, 1, 2 , 3 money units per period with probability $Prob_m$ where

$Prob_0 := .25$ $Prob_1 := .25$ $Prob_2 := .25$ $Prob_3 := .25$

Assume that he consumes a quarter of his wealth each period. The transition law is approximated by rounding the consumption (cons) to the nearest integer:

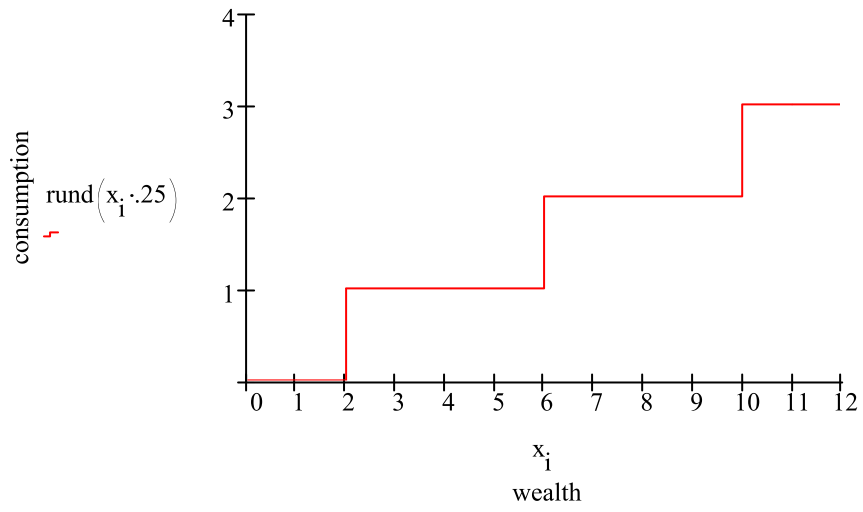
$rund(cons) := wenn(cons - floor(cons) < .5, floor(cons), ceil(cons))$

Defining the possible **states of wealth** as

$i := 0 .. 12$

$x_i := i$

we obtain the following individual **consumption function**:



The **transition matrix** must be:

$$
P := \left| \begin{array}{l} c \leftarrow .25 \\ \text{for } i \in 0 .. 12 \\ \quad \text{for } j \in 0 .. 12 \\ \quad\quad \left| \begin{array}{l} P_{i,j} \leftarrow Prob_0 \ \text{ if } i - rund(i \cdot c) = j \\ P_{i,j} \leftarrow Prob_1 \ \text{ if } i + 1 - rund(i \cdot c) = j \\ P_{i,j} \leftarrow Prob_2 \ \text{ if } i + 2 - rund(i \cdot c) = j \\ P_{i,j} \leftarrow Prob_3 \ \text{ if } i + 3 - rund(i \cdot c) = j \\ P_{i,j} \leftarrow 0 \ \text{ otherwise} \end{array} \right. \\ P \end{array} \right.
$$

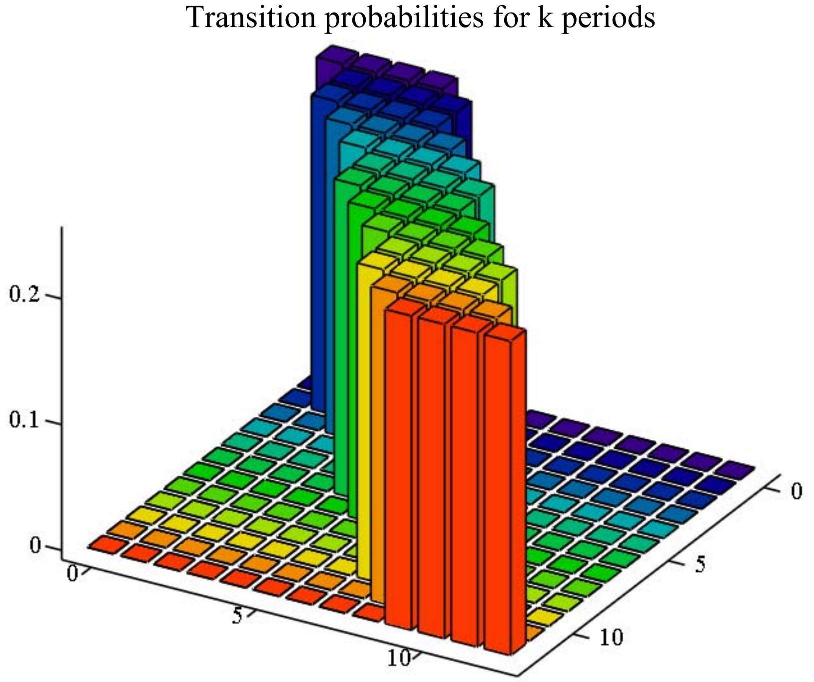
As an **initial distribution** of x we use for example $\pi 0_{0,i} := \frac{1}{13}$.

Let's check the validity of our model:

$\text{validity}\left(P,\pi_0\right) = \text{"O.K."}$

Iterate the transition matrix:

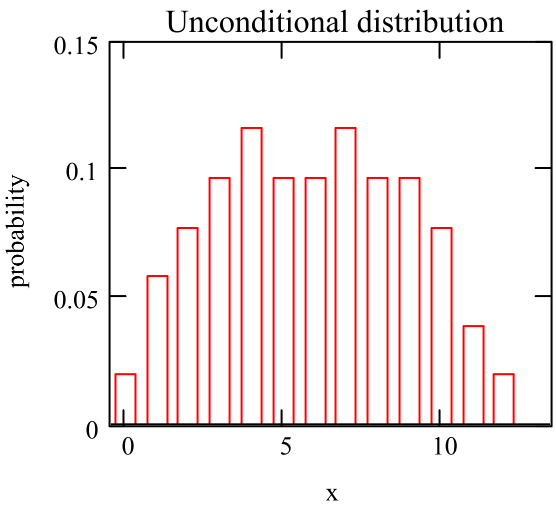
$k := 1$



P^k

Do the same for the unconditional probabilities to approximate the stationary distribution:

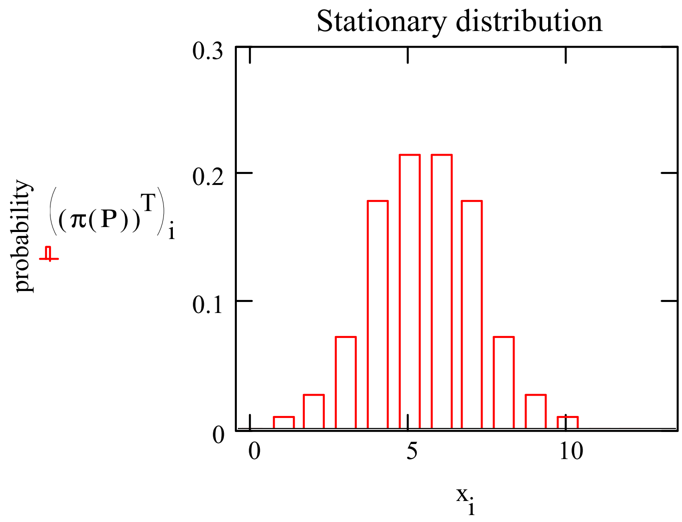
$k := 1$



with the (unconditional) **expectation:**

$\pi_0 \cdot P^k \cdot x = (5.8846154)$

Here follows the direct way to compute the stationary distribution:

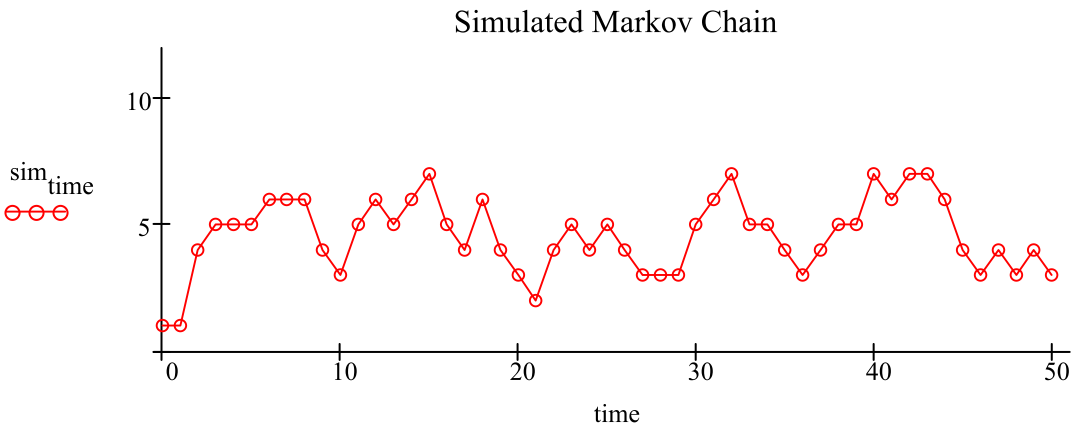


At last we simulate the wealth of this individual over time:

$k := 50$

$sim := rdmarkov(k, P, \pi_0, x)$ ⇐ Click with your mouse on the red field and press the F9-button to get another random selection!

$time := 0 .. k$



Try another probability distribution $Prob_m$ of income!

Literature:

Judd, K.L.: Numerical Methods in Economics. Cambridge (MA)/London 1998, p. 85 - 84.