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Monopoly

Part III: Multiple Local Equilibria and Adaptive Search

Summary:

The information about market demand needed by a profit maximizing monopolist is rather more complex and costly than that needed by a "price taker" in a perfect competitive market: The monopolist must know at least the entire demand curve instead of just one point on it. In text book cases the demand curve is a simple downward sloping straight line. But in her pioneering work Joan Robinson 1933 has already remarked that "cases of multiple equilibrium may arise when the demand curve changes its slope, being highly elastic for a stretch, then perhaps becoming relatively inelastic, then elastic again". This may happen if the demand is a composite of the demand from several subgroups of consumers, with different elasticities and specific reservation prices. Puu (1997) further assumed, that the monopolist doesn't know the demand function. Hunting for a profit maximum, the firm follows an adaptive search procedure just using the information about recent reaction of profits to supply changes. Such an output rule may cause periodic cycles and chaotic behaviour of the monopolistic supply. This worksheet allows you to explore the complex dynamics of Puu's model in depth.

I. The case with one product

1. Basic assumptions

The (inverse) **demand function** is

$$p(x) := \alpha_0 - \alpha_1 \cdot x + \alpha_2 \cdot x^2 - \alpha_3 \cdot x^3$$

where p denotes commodity price, x denotes quantity demanded and α_i represents (positive) constants:

$$\alpha_0 \equiv \frac{56}{10} \qquad \alpha_1 \equiv \frac{27}{10} \qquad \alpha_2 \equiv \frac{62}{100} \qquad \alpha_3 \equiv \frac{1}{20}$$

This function is down-sloping and thus invertible provided that the following condition holds (=1):

$$\left(\alpha_2^{\,2} < 3 \cdot \alpha_1 \cdot \alpha_3\right) = 1$$

Because the total revenue is R(x) = p(x)·x, the **marginal revenue** must be:

$$R'(x) := \frac{d}{dx} p(x) \cdot x \text{ vereinfachen } \rightarrow \frac{-27}{5} \cdot x + \frac{93}{50} \cdot x^2 - \frac{1}{5} \cdot x^3 + \frac{28}{5}$$

To provide an opportunity of multiple equilibria, the condition for an upward slope at the point of inflection of the marginal revenue curve must hold (=1):

$$\left(\frac{8}{3} \cdot \alpha_1 \cdot \alpha_3 < \alpha_2^{\,2}\right) = 1$$

Next we define the **total cost** as a function of the quantity supplied:

$$K(x) := \beta_0 + \beta_1 \cdot x - \beta_2 \cdot x^2 + \beta_3 \cdot x^3$$

As positive constants we take:

$$\beta_0 \equiv 0 \qquad \beta_1 \equiv 2 \qquad \beta_2 \equiv \frac{3}{10} \qquad \beta_3 \equiv \frac{1}{50}$$

Therefore, the **marginal cost** dK/dx must be:

$$K'(x) := \frac{d}{dx} K(x) \rightarrow 2 - \frac{3}{5} \cdot x + \frac{3}{50} \cdot x^2$$

The **profit function** is computed by

$$\Psi(x) := p(x) \cdot x - K(x) \text{ vereinfachen } \rightarrow \frac{18}{5} \cdot x - \frac{12}{5} \cdot x^2 + \frac{3}{5} \cdot x^3 - \frac{1}{20} \cdot x^4$$

The **first order condition (FOC)** of a profit maximum is $\Psi'(x)=0$. We determine the solution $x=x_{FOC}$ for this equation:

1. Compute the marginal profit:

$$\Psi'(x) := \frac{d}{dx}\Psi(x) \rightarrow \frac{18}{5} - \frac{24}{5}\cdot x + \frac{9}{5}\cdot x^2 - \frac{1}{5}\cdot x^3$$

2. Collect the coefficients of this polynom:

$$\text{coeff} := \Psi'(x) \text{ koeff}, x \rightarrow \begin{bmatrix} \frac{18}{5} \\ \frac{-24}{5} \\ \frac{9}{5} \\ \frac{-1}{5} \end{bmatrix}$$

3. Solve the polynom:

$$x_{FOC} := \text{nullstellen(coeff)} \qquad x_{FOC} = \begin{bmatrix} 1.268 \\ 3 \\ 4.732 \end{bmatrix} \qquad \overrightarrow{\Psi(x_{FOC})} = \begin{bmatrix} 1.8 \\ 1.35 \\ 1.8 \end{bmatrix}$$

Let's check for the **second order condition**:

1. Compute the second derivative of the profit function:

$$\Psi''(x) := \frac{d^2}{dx^2}\Psi(x) \rightarrow \frac{-24}{5} + \frac{18}{5}\cdot x - \frac{3}{5}\cdot x^2$$

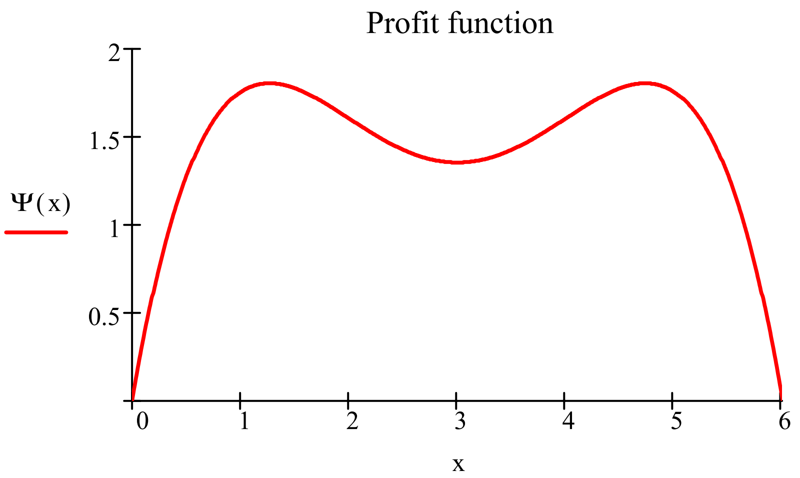
2. Search for local maxima where $\Psi''(x_{FOC})<0$:

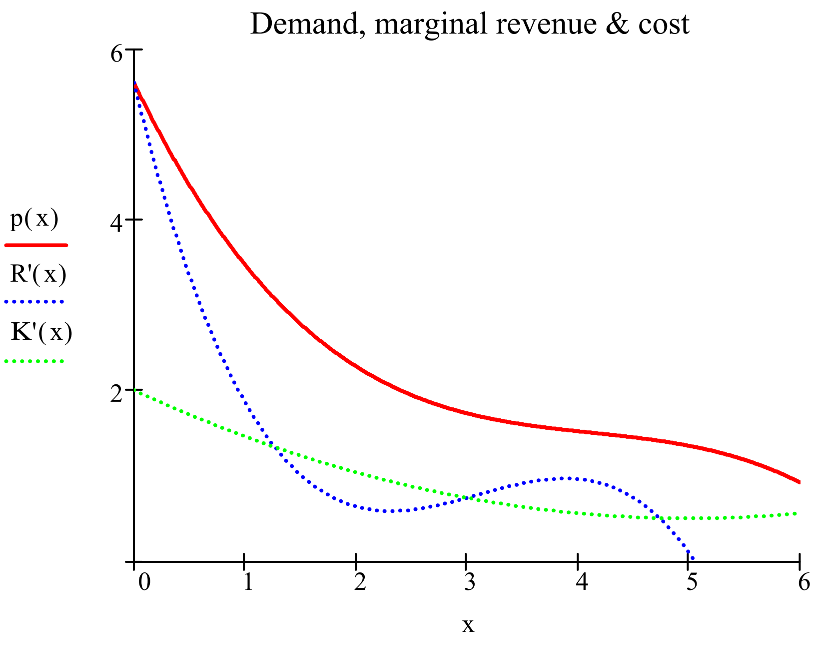
$$\overrightarrow{\Psi''(x_{FOC})} = \begin{bmatrix} -1.2 \\ 0.6 \\ -1.2 \end{bmatrix}$$

Now take a look to the figures below to verify the results from above:

$xmax := 6$

$x := 0, \frac{xmax}{1000} .. xmax$





2. Introducing adaptive search

Assume that the monopolist doesn't know more than a few points on the demand function. This information would be of local character and short lifetime.The monopolist might not even know that globally there are two distinct profit maxima. Given this, the monopolist follows a simple search algorithm (in the vein of Newton) for the maximum of the unknown profit function: He estimates the difference of profits from the last two visited points x_{time} and x_{time-1}. Then he uses a given step length $\delta > 0$ to move his supply in the direction of increasing profits:

step size: $\delta := 1.6$

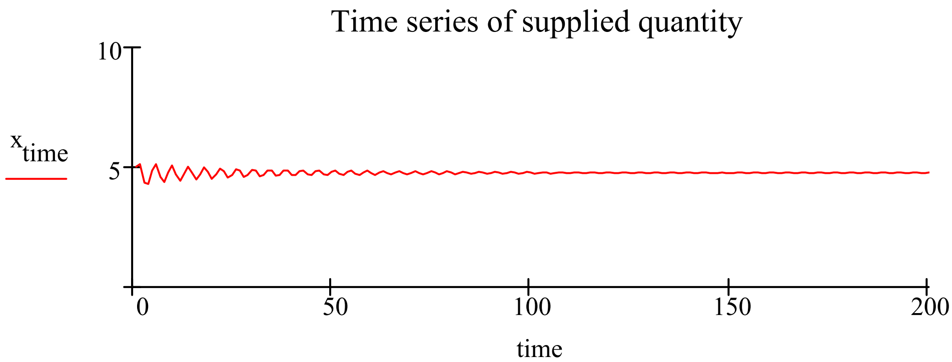
initial values: $x_1 := 5$ $x_2 := 5.1$

maximum of periods computed: $T := 200$

$time := 3..T$

$$x_{time} := x_{time-1} + \delta \cdot \frac{\Psi\left(x_{time-1}\right) - \Psi\left(x_{time-2}\right)}{x_{time-1} - x_{time-2}}$$

$time := 1..T$

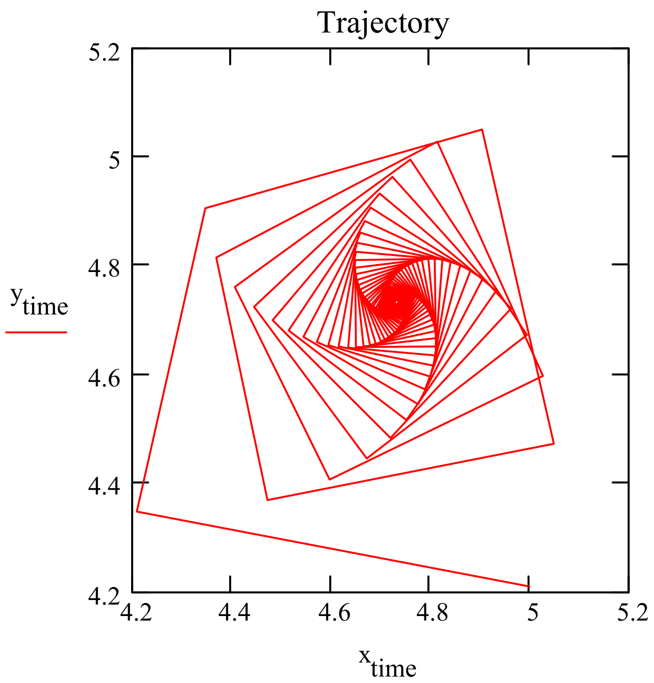


The search procedure from above describes a difference equation of second order. We can decompose this in a system of two first order equations by defining:

$y_2 := x_1$ $y_3 := x_2$ $time := 3..T$

$$\begin{bmatrix} x_{time} \\ y_{time} \end{bmatrix} := \begin{bmatrix} y_{time-1} \\ y_{time-1} + \delta \cdot \dfrac{\Psi\left(y_{time-1}\right) - \Psi\left(x_{time-1}\right)}{y_{time-1} - x_{time-1}} \end{bmatrix}$$

Hence, we draw the trajectory for this iterated map:



3. Fixed points, cycles and chaos

Not unexpected, the maxima and the minimum of the profit function are fixed points of the iterated map given by

$x_{time} = y_{time}$ and $\Psi'(x_{time}) = 0$

Of course the minimum must be an unstable fixed point. To find out the stability of the maxima we reformulate the iterated map to use *Mathcad*'s symbolic processor:

$$f(X,Y) := Y$$

$$g(X,Y) := Y + \delta \cdot \frac{\Psi(Y) - \Psi(X)}{Y - X}$$

Now we are able to evaluate the Jacobian:

$$\text{Jacobian}(X) := \left| \begin{bmatrix} \frac{d}{dX}f(X,Y) & \frac{d}{dY}f(X,Y) \\ \frac{d}{dX}g(X,Y) & \frac{d}{dY}g(X,Y) \end{bmatrix} \right| \begin{array}{|l} \text{ersetzen}, Y = X \\ \text{vereinfachen} \end{array} \rightarrow \frac{3}{10} \cdot \delta \cdot \left(X^2 - 6 \cdot X + 8\right)$$

Loss of stability occurs when the Jacobian is unitary. Thus, we get as the critical value δ_c of the step width:

$$\delta_c(X) := \text{Jacobian}(X) = 1 \text{ auflösen}, \delta \rightarrow \frac{10}{\left[3\cdot\left(X^2 - 6\cdot X + 8\right)\right]}$$

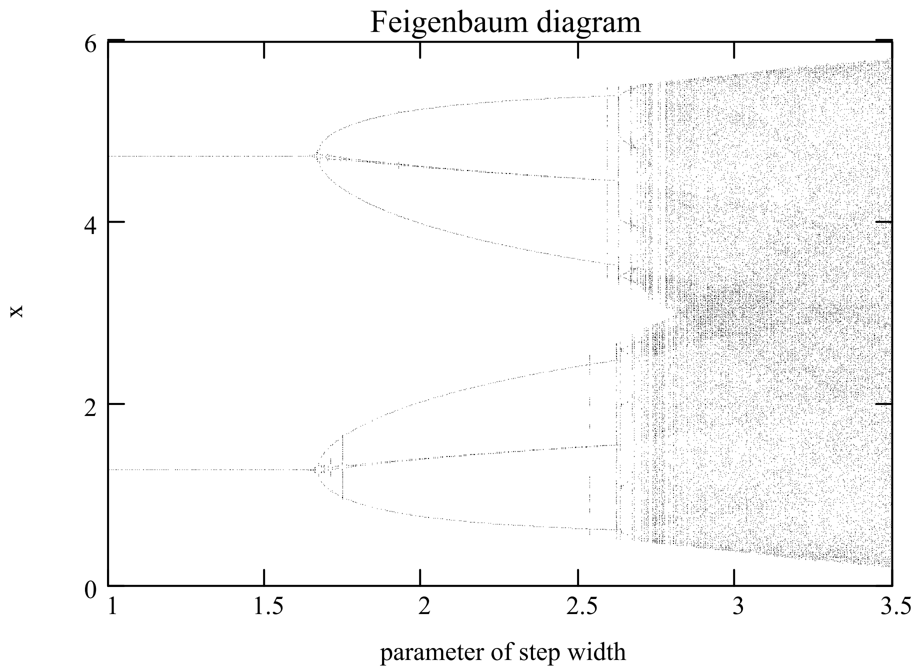
$$\overrightarrow{\delta_c\left(x_{FOC}\right)} = \begin{bmatrix} 1.666667 \\ -3.333333 \\ 1.666667 \end{bmatrix}$$

Plotting the phase variable x against the parameter δ in a Feigenbaum diagram, we see the alternative fixed points coexist at $\delta < \delta_c = \frac{5}{3}$. After that they are replaced by cycles. (Note: This cycles are of period 4 with one value taken on twice as often as are the other two!) Chaos takes over once this cycles lose stability. Puu (1997, p. 123-126) proves that this happens if the value of the step parameter exceeds 2.48813. At first there appears chaotic bands, with each its own basin of attraction. After a certain point the attractors merge in a single one. To run the bifurcation plot you must enter a resolution number RES. (Note: High resolution is very time consuming.)

Resolution (RES= 1,2, ...,10): $RES \equiv 10$

range of δ: $\delta_{bot} \equiv 1$ $\delta_{top} \equiv 3.5$

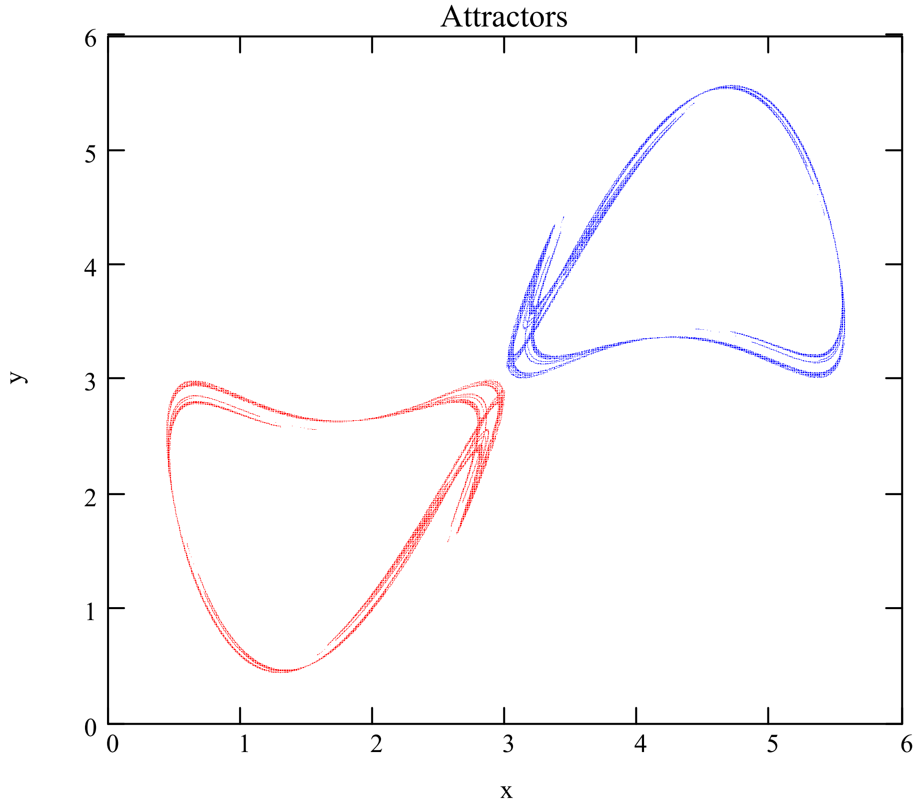
range of x: $x_{bot} \equiv 0$ $x_{top} \equiv 6$



4. Attractors

The next figure plots attractors of this system into the phase plane. Change δ using the different values from above to see fixed points, perodic cycles and two coexistent chaotic attractors merging to a single one. Note the symmetry of the shapes of the attractors. For symmetric systems any attractors that are not symmetric in themselves come pairwise, forming together a symmetric picture.

$\delta := 2.8$



II. The case with two complementary products

Suppose the monopolist supplies two goods x and y that are complementary in consumers perferences. Both goods have demand functions similar to the one in case I. But now the products of sales volumes $x^2 \cdot y$ respectively $x \cdot y^2$ enter as a positive term in both demand functions. This models the "coupling effect" due to complementary. Puu uses the same cost function K(x+y) from above, stating the modified profit function Π(x,y) with the "coupling" parameter κ as:

$$\Pi(x,y,\kappa) := \left[\Psi(x) + \Psi(y) + \kappa \cdot \left(x^2 \cdot y + x \cdot y^2\right)\right]$$

Again it will be assumed, that the firm hunts with an adaptive search procedure for the profit maximum. To simulate this behaviour we need the gradient of profits:

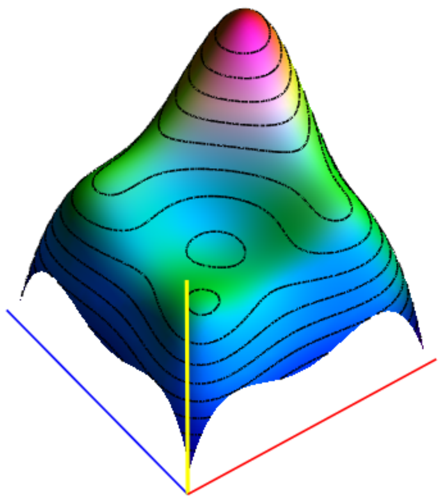
$$\Pi_x(x,y,\kappa) := \frac{d}{dx}\Pi(x,y,\kappa) \rightarrow \frac{18}{5} - \frac{24}{5} \cdot x + \frac{9}{5} \cdot x^2 - \frac{1}{5} \cdot x^3 + \kappa \cdot \left(2 \cdot x \cdot y + y^2\right)$$

$$\Pi_y(x,y,\kappa) := \frac{d}{dy}\Pi(x,y,\kappa) \rightarrow \frac{18}{5} - \frac{24}{5} \cdot y + \frac{9}{5} \cdot y^2 - \frac{1}{5} \cdot y^3 + \kappa \cdot \left(x^2 + 2 \cdot x \cdot y\right)$$

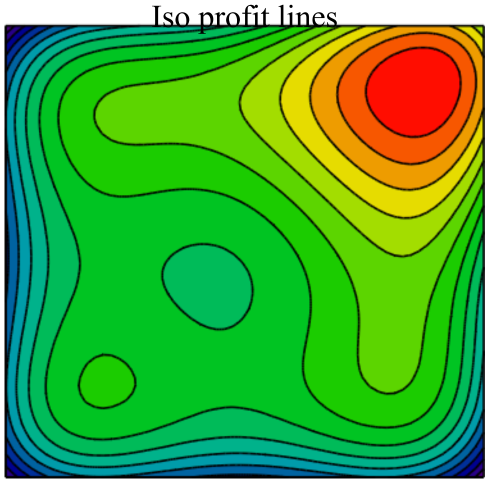
Given a small value of κ (for example 0.0015) we obtain a profit function with four different local maxima:

$\kappa \equiv 0.01$

Profit function



profit



profit

To start the adaptive search procedure we set

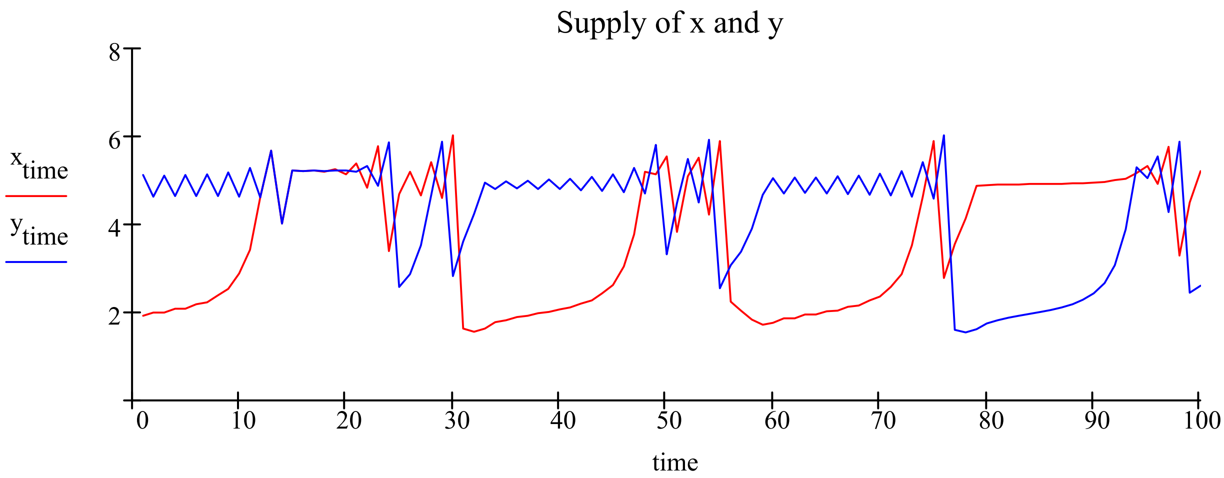
step size: $\delta := 1.37$ **initial values:** $x_1 := 1.9$ $y_1 := 5.1$

maximum of periods computed: $T := 50000$

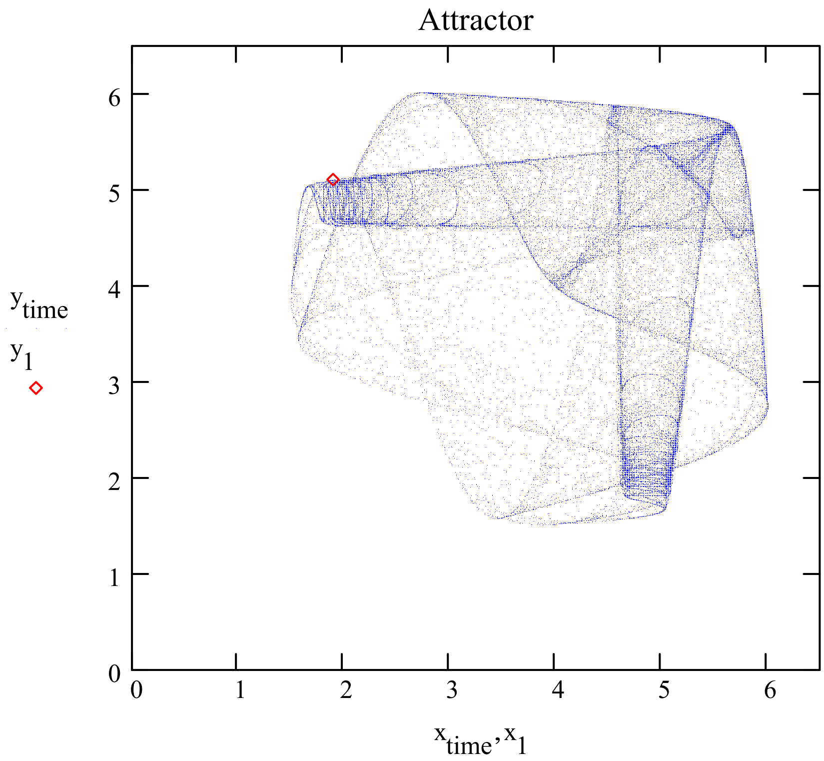
$time := 2 .. T$

$$\begin{bmatrix} x_{time} \\ y_{time} \end{bmatrix} := \begin{bmatrix} x_{time-1} + \delta \cdot \Pi_x\left(x_{time-1}, y_{time-1}, \kappa\right) \\ y_{time-1} + \delta \cdot \Pi_y\left(x_{time-1}, y_{time-1}, \kappa\right) \end{bmatrix}$$

$time := 1 .. 100$



$t_{start} := 1000$ $time := t_{start} .. T$



Now run this system with increasing step length δ. First you will find coexistent stable fixed points. After their loss of stability these fixed points are replaced by periodic cycles and then chaotic attractors. You will also find coexistent cycles and chaotic attractors at once. It depends on the initial conditions embedded in different basins of attraction, to which one the process settles. Here are some examples of parameter combinations you should try (initial conditions are always $x_1=1.9$ and $y_1=5.1$):

- Case I: Lower coupling of the products with $\kappa=0.0015$

$\delta=1.5$	$\Rightarrow$	fixed point in NW
$\delta=2$	$\Rightarrow$	cycle with period 2 in NW
$\delta=2.1$	$\Rightarrow$	period doubling
$\delta=2.3$	$\Rightarrow$	chaotic attractor ("two stripes" in NW)
$\delta=2.5$	$\Rightarrow$	chaotic attractor ("product set" in NW)
$\delta=2.6$	$\Rightarrow$	chaotic attractor ("product set" in SW)
$\delta=2.7$	$\Rightarrow$	chaotic attractor (expansion of the "product set" with sparse points)
$\delta=2.8$	$\Rightarrow$	chaotic attractor (expanded "product set" with sparse points only in NW)

- Case II: Medium coupling of the products with $\kappa=0.006$

$\delta=2$	$\Rightarrow$	chaotic "Henon"-like attractor in NW
$\delta=2.4$	$\Rightarrow$	periodic cycles in SW
$\delta=2.5$	$\Rightarrow$	chaotic attractor in SW
$\delta=2.7$	$\Rightarrow$	chaotic attractor from SW expanding to SE and NW

- Case III: Strong coupling of the products with $\kappa=0.01$

$\delta=1.37$	$\Rightarrow$	strange attractor in NE
$\delta=1.4$	$\Rightarrow$	fixed point in SW

Literature:

Puu, T.: Nonlinear Economic Dynamics. Berlin/Heidelberg 1997, 113 - 131.

Robinson, J.: The Economics of Imperfect Competition. 2nd ed., London 1969.

$ORIGIN \equiv 1$