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A Simple Exchange Economy with Complex Dynamics

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Summary:

Mukherji (1999) shows that a standard discrete tatonnement process within the context of a very simple exchange economy (two goods, two persons with Cobb-Douglas utility functions) exhibits complex dynamics of the price adjustment. This worksheet gives you the numerical tools to explore the phenomenon of period doubling bifurcation and chaos in this model.

1. Important definitions

A, B:	individuals
p:	price of good x relative to good y
p_E:	equilibrium price
p_x, p_y:	absolute prices of x and y
u_A, u_B:	utility of individual A and B
x, y:	quantities of goods
x_o, y_o:	endowments of goods
Z:	excess demand of good x

2. Basic assumptions

The preferences of individual A are given by:

$u_A(x, y, \alpha) := x^{\alpha} \cdot y^{1-\alpha}$ with $0 < \alpha < 1$

The preferences of individual B are given by:

$u_B(x, y, \beta) := x^{\beta} \cdot y^{1-\beta}$ with $0 < \beta < 1$

We define the price of good y as a numeraire:

$p_y = 1$

Thus the relative price is written as:

$$p = \frac{p_x}{p_y} = p_x$$

Individual A possesses the endowment $(x_o, 0)$ and B has the endowment $(0, y_o)$. Therefore, the budget constraints are

$x_o \cdot p \geq x \cdot p + y$ for individual A

$y_o \geq x \cdot p + y$ for individual B

3. The exchange equilibrium

With these budget constraints the **first order conditions of utility maximization** yield the demand functions for good x of individual A and B:

$$x_A(p,\alpha,x_o) := \frac{d}{dx}u_A(x, x_o \cdot p - x \cdot p, \alpha) \text{ auflösen}, x \rightarrow \alpha \cdot x_o$$

$$x_B(p,\beta,y_o) := \frac{d}{dx}u_B(x, y_o - x \cdot p, \beta) \text{ auflösen}, x \rightarrow \beta \cdot \frac{y_o}{p}$$

Now we summarize the demand behaviour by the **excess demand function** Z (·) for good x:

$$Z(p,\beta,\alpha,x_o,y_o) := \left(x_B(p,\beta,y_o) + x_A(p,\alpha,x_o) - x_o\right) \rightarrow \beta \cdot \frac{y_o}{p} + \alpha \cdot x_o - x_o$$

The market is in equilibrium if Z(·) = 0. Hence the unique **equilibrium price** is determined by:

$$p_E(\beta,\alpha,x_o,y_o) := Z(p,\beta,\alpha,x_o,y_o) \text{ auflösen}, p \rightarrow -\beta \cdot \frac{y_o}{\left[x_o \cdot (-1+\alpha)\right]}$$

4. Introducing adjustment dynamics

Consider the standard **adjustment on prices in disequilibrium** (the "tatonnement")

$$p_{i+1} = p_i + \gamma \cdot Z(p_i, \beta, \alpha, x_o, y_o)$$

where $\gamma>0$ is some constant **speed of adjustment**.We can rewrite this equation as an iterated map:

$$f(p) := \left(p + \gamma \cdot Z(p,\beta,\alpha,x_o,y_o)\right) \rightarrow p + \gamma \cdot \left(\beta \cdot \frac{y_o}{p} + \alpha \cdot x_o - x_o\right)$$

First order conditon gives:

$$\frac{d}{dp}f(p) \text{ auflösen}, p \rightarrow \begin{bmatrix} (\gamma \cdot \beta \cdot y_o)^{\left(\frac{1}{2}\right)} \\ -(\gamma \cdot \beta \cdot y_o)^{\left(\frac{1}{2}\right)} \end{bmatrix}$$

Insert the positive solution into the **second order condition**:

$$\frac{d^2}{dp^2} f(p) \text{ ersetzen}, p = \sqrt{\gamma \cdot \beta \cdot y_o} \rightarrow 2 \cdot \gamma \cdot \beta \cdot \frac{y_o}{\left(\gamma \cdot \beta \cdot y_o\right)^{\left(\frac{3}{2}\right)}}$$

Because the second derivate becomes positive we know that f(p) attains a **minimum** value at

$$p' = \sqrt{\gamma \cdot \beta \cdot y_o}$$

given by

$$f\left(\sqrt{\gamma \cdot \beta \cdot y_o}\right) \text{ vereinfachen } \rightarrow \gamma \cdot \frac{\left[2 \cdot \beta \cdot y_o - x_o \cdot \left(\gamma \cdot \beta \cdot y_o\right)^{\left(\frac{1}{2}\right)} + \alpha \cdot x_o \cdot \left(\gamma \cdot \beta \cdot y_o\right)^{\left(\frac{1}{2}\right)}\right]}{\left(\gamma \cdot \beta \cdot y_o\right)^{\left(\frac{1}{2}\right)}}$$

... or more simplified (by hand and not by *Mathcad*):

$$f(p') = 2 \cdot \sqrt{\gamma \cdot \beta \cdot y_o} - \gamma \cdot (1 - \alpha) \cdot x_0$$

In order to guarantee positive prices $f(p') > 0$ must hold. Defining

$$K = \frac{\gamma \cdot \left[(1 - \alpha) \cdot x_o\right]^2}{\beta \cdot y_o}$$

this is ensured if $K < 4$.

5. Some properties of the adjustment dynamics

For the proofs of the following cited claims, see Mukherij (1999).

Claim 2: $K<2 \Rightarrow p_E$ is locally stable for the process f(p).

Claim 3: For $2<K<2.5$ there exists a stable 2-cycle.

Let K_n denote the critical value of K where a 2^n cycle is born; then $K_1 := 2$ and $K_2 := 2.5$.

Using the Feigenbaum constant $F_{const} := 4.6692016091029$ the value of $\kappa = \lim_{n \to \infty} K_n$

can be approximated by:

$$\kappa := \frac{F_{const} \cdot K_2 - K_1}{F_{const} - 1} \qquad \Rightarrow \qquad \kappa = 2.636$$

Claim 4: For $K \in \Delta = (3.0, 3.6)$ the map f(p) exhibits topological chaos.

Claim 5: For $K = 25/9$, the map f(p) exhibits ergodic chaos; in addition there exists $K \in \Delta$ such that f(p) exhibits ergodic chaos.

6. Numerical Explorations

To explore the behaviour of the attractors for different values of K, Mukherji (1999, p.745) fixes the values of all parameters except the adjustment coefficient γ with

$\beta \cdot y_o = 1$ and $(1 - \alpha) \cdot x_o = 6$

so that $K = 36 \cdot \gamma$. Then the iterated map takes the particular form:

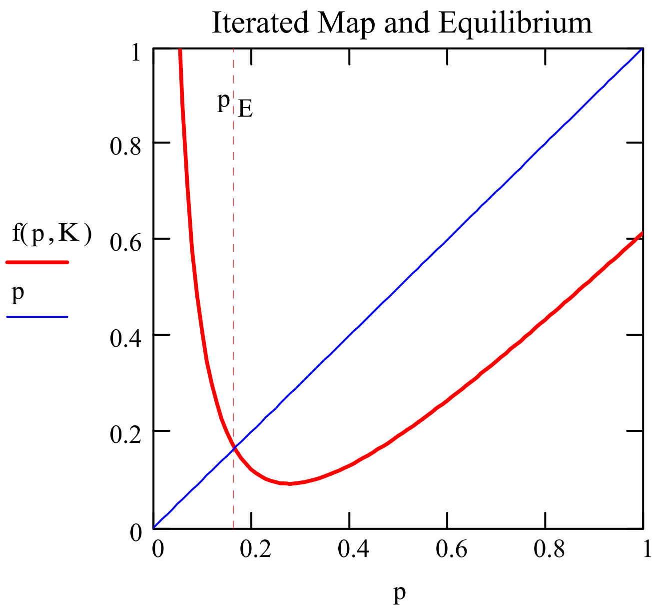
$$f(p, K) := p + \left(\frac{1}{p} - 6\right) \frac{K}{36}$$

Notice, that under these parameter restrictions the value of the equilibrium price is independent from K:

$p_E := f(p, K) = p$ auflösen, p $\rightarrow \frac{1}{6}$

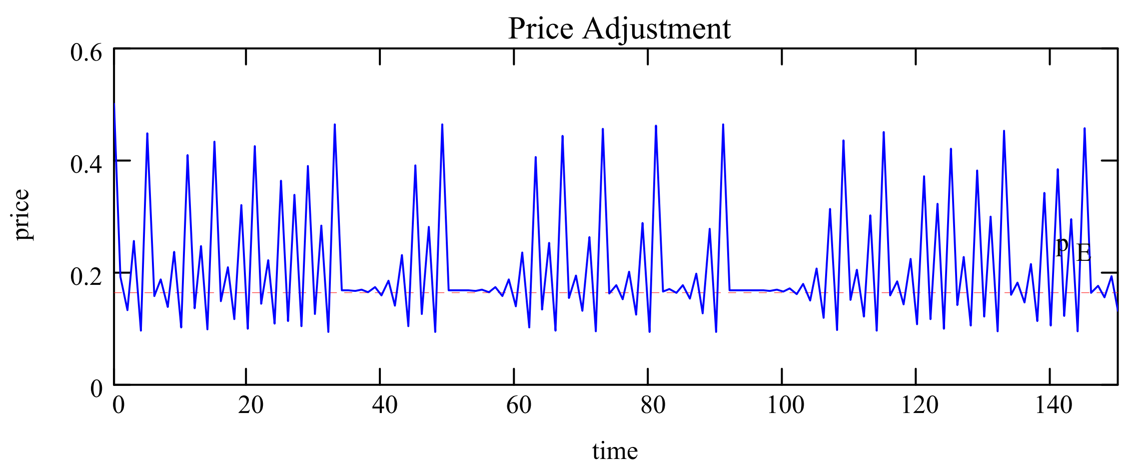
Choosing a numerical value for K, the iterated map is drawn in the following figure:

$K := \frac{25}{9}$ $\quad$ $p_{max} := 1$ $\quad$ $p := 0, .01 .. p_{max}$

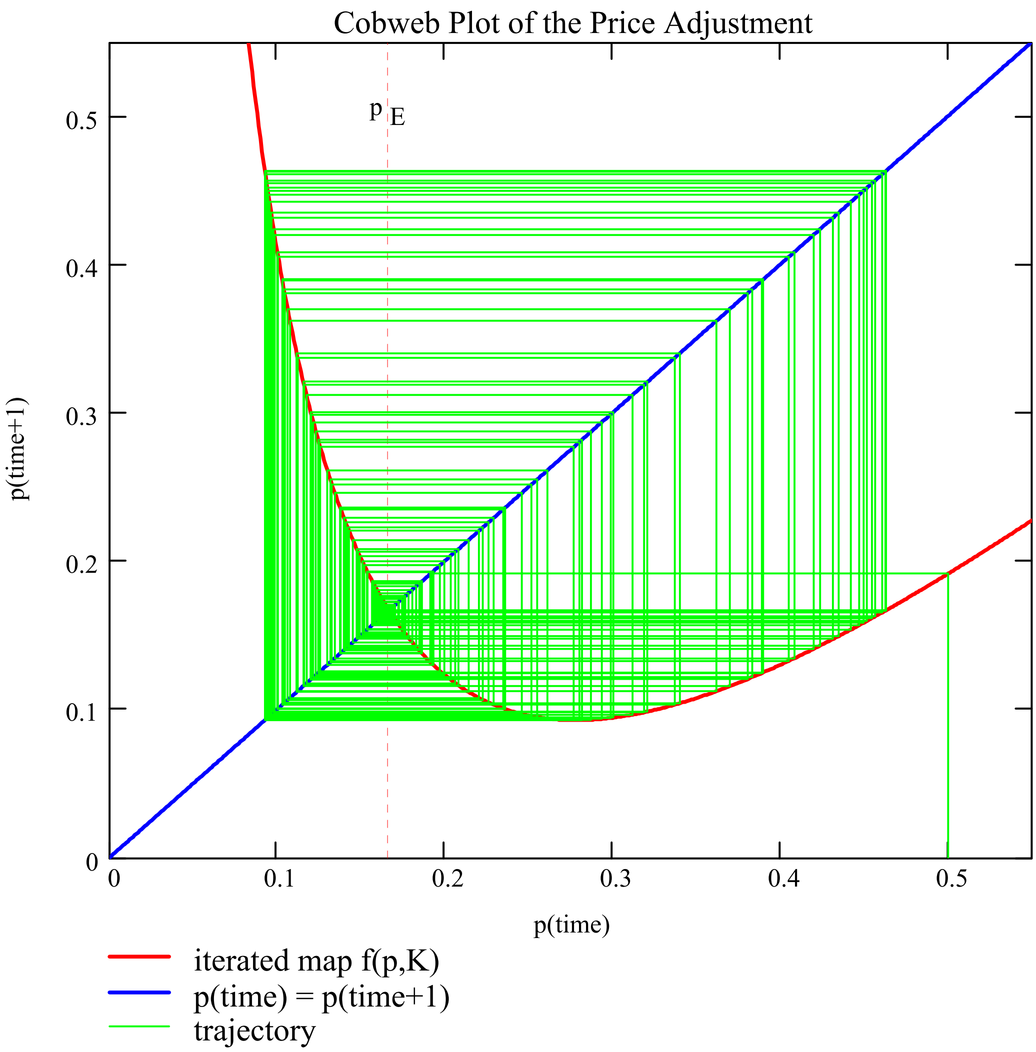


Given an initial value p_0 of the first price offer and using the K-value from above, the adjustment process is computed for T_{max} periods .

$p_0 := .5$ $\quad$ $T_{max} := 150$ $\quad$ $i := 0 .. T_{max}$ $\quad$ $p_{i+1} := f(p_i, K)$



This adjustment process may be also presented by a cobweb plot.

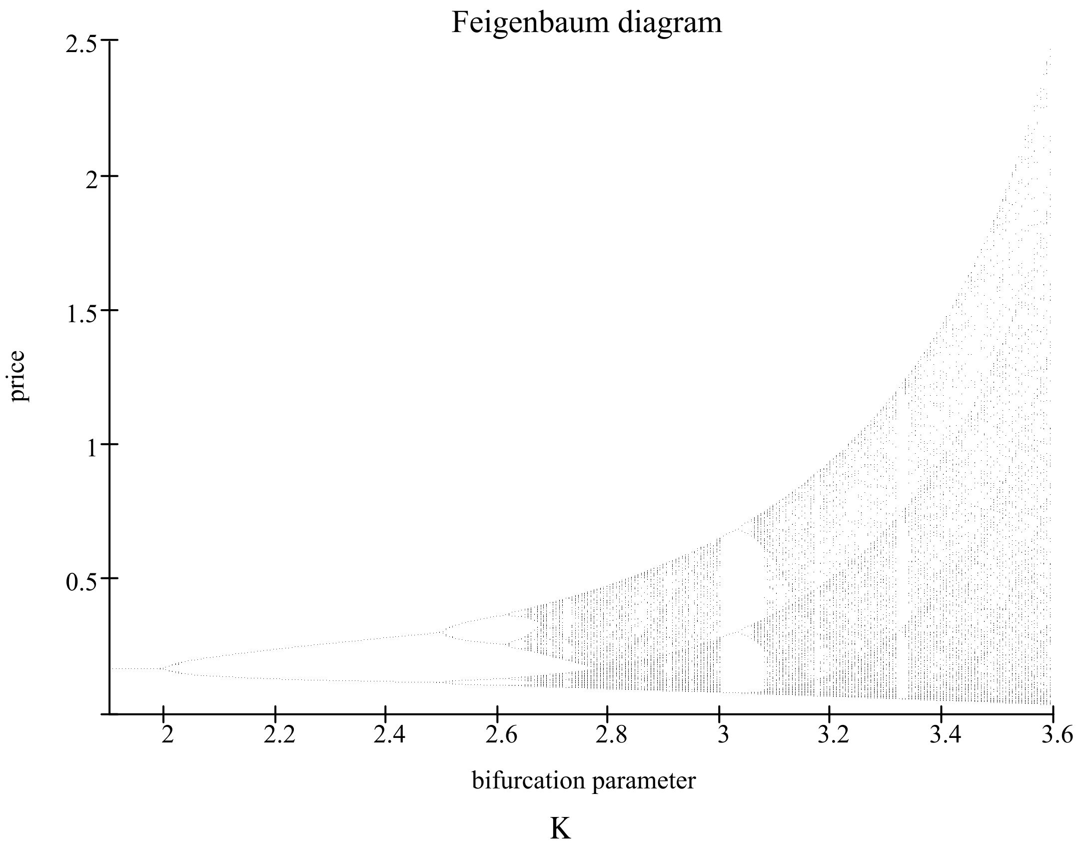


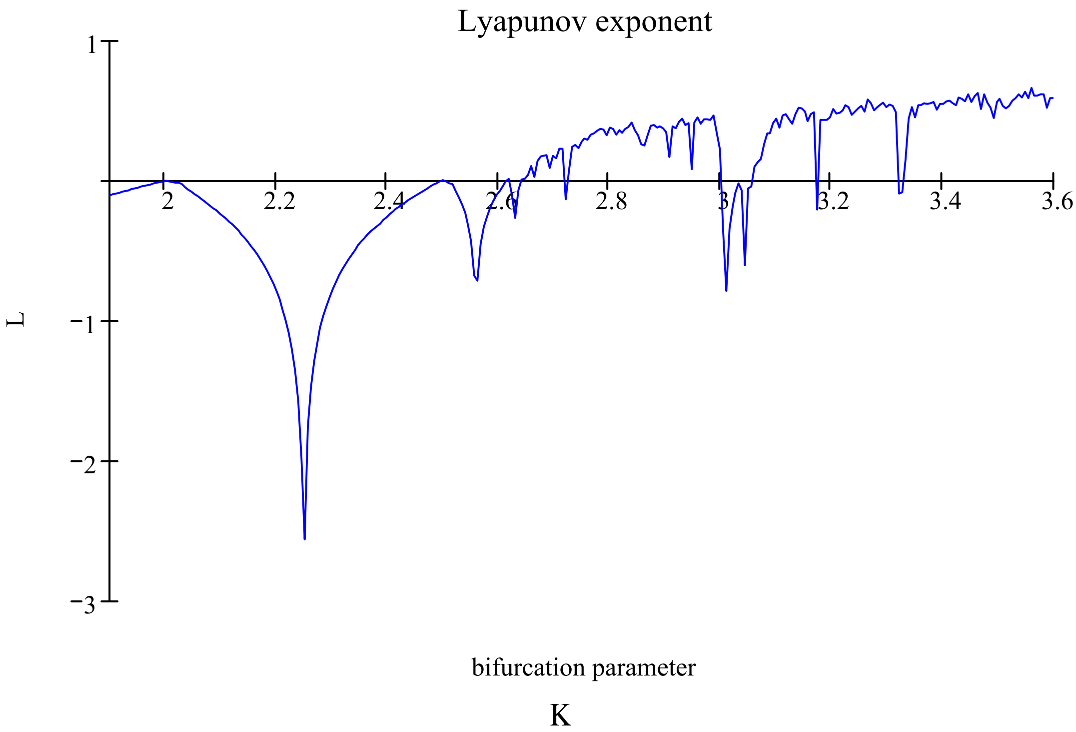
Observe the bifurcations by plotting the **Feigenbaum diagram** and the **Lyapunov exponent**. Use these figures to choose K such that you obtain stable cycles of different periodicity or irregular cycles of p.

Resolution of graph: $RES := 3$ $(1, 2, .., 10)$

Range of plotted values: $K_{bottom} := 1.9$ $K_{top} := 3.6$

$p_{bottom} := 0$ $p_{top} := 2.5$





Note: Positive values of the Lyapunov exponent indicate chaotic behaviour of p!

Literature:

Anjan Mukherji: A Simple Example of Complex Dynamics. In: Economic Theory, vol.14 (1999), 741 - 749.