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Modeling Economic Time Series with Stochastic Linear Difference Equations

Summary:

Hansen/Sargent (1998) created a class of models that merges recursive linear models of dynamic economies with dynamic econometrics. The underlying stochastic process of the vector of economic state variables is constructed recursively using an initial random vector and a time invariant law of motion. Some simulations will show how special cases of this general model are formed by a variety of time series processes that have been studied by economists.

The basic law of motion equation:

$$x^{\langle t+1\rangle} = A \cdot x^{\langle t\rangle} + C \cdot w^{\langle t+1\rangle}$$

$x^{\langle t\rangle}$: sequence of random vectors

$w^{\langle t\rangle}$: sequence of "white-noise"-vectors (martingale difference sequence)

A , C : transition matrices

Example 1: Deterministic polynomial time trends

Trend function:

$$y_t = \Delta_1 + \Delta_2 \cdot t + \Delta_3 \cdot t \cdot (t-1) + \Delta_4 \cdot t \cdot (t-1) \cdot (t-2)$$

Without random effects C = 0 and w = 0. The transition matrix A becomes

$$A := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence $x^{\langle t+1\rangle} = A \cdot x^{\langle t\rangle} = A^t \cdot x^{\langle 0\rangle}$ with

$$A^t = \begin{bmatrix} 1 & t & \frac{t\cdot(t-1)}{2} & \frac{t\cdot(t-1)\cdot(t-2)}{6} \\ 0 & 1 & t & \frac{t\cdot(t-1)}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Given the initial condition

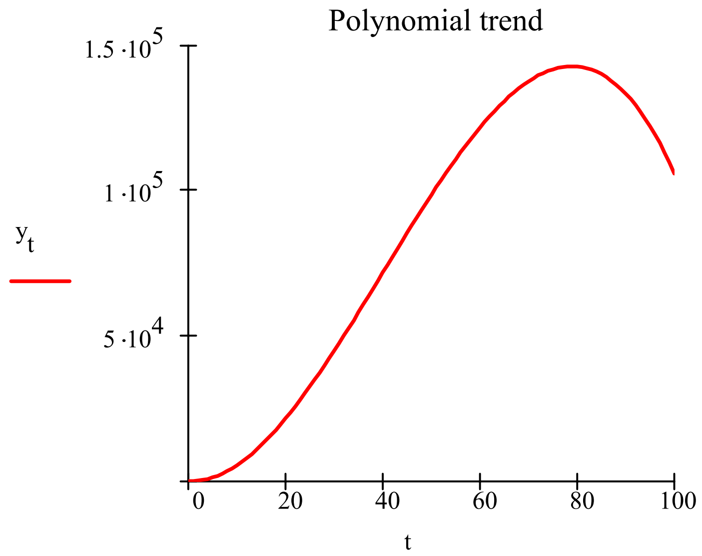
$$x^{<0>} := \begin{bmatrix} \Delta_1 & \Delta_2 & 2\cdot\Delta_3 & 6\cdot\Delta_4 \end{bmatrix}^T$$

we obtain for $t := 1 .. 100$ the state vectors

$$x^{<t>} := A \cdot x^{<t-1>}$$

The first element of x yields the cubic polynom of the trend function above. If you insert 0 to the other elements of $x^{<0>}$, you will get a linear or quadratic trend function.

$t := 0 .. 100 \qquad y_t := \left(x^{<t>}\right)_0$



Polynomial coefficients:

$\Delta_1 \equiv 80$

$\Delta_2 \equiv -50$

$\Delta_3 \equiv 70$

$\Delta_4 \equiv -0.6$

Example 2: Deterministic seasonals

To represent the model

$y_t = y_{t-4}$

let n = 4, C = 0, $A := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and $x^{<t>} = \left(y_t, y_{t-1}, y_{t-2}, y_{t-3}\right)^T$

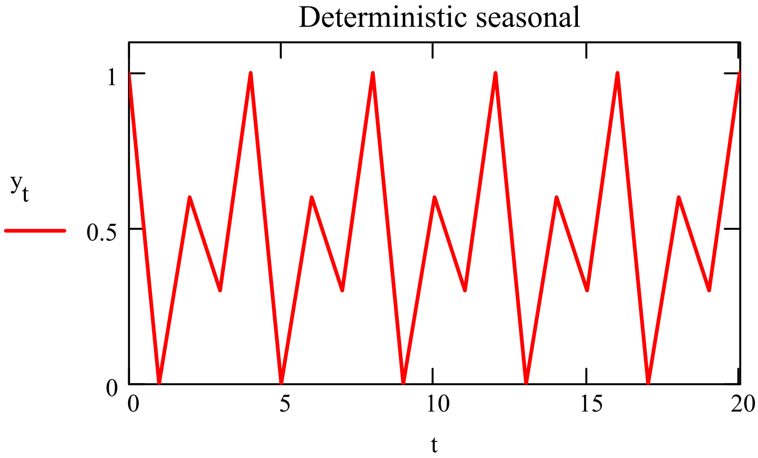
Given a seasonal pattern to the initial vector

$x^{<0>} := \left[S_1 \;\; S_2 \;\; S_3 \;\; S_4\right]^T$,

the system is completed:

$t := 1..20 \qquad x^{<t>} := A \cdot x^{<t-1>}$

$t := 0..20 \qquad y_t := \left(x^{<t>}\right)_0$



Seasonal pattern:

$S_1 \equiv 1$

$S_2 \equiv .3$

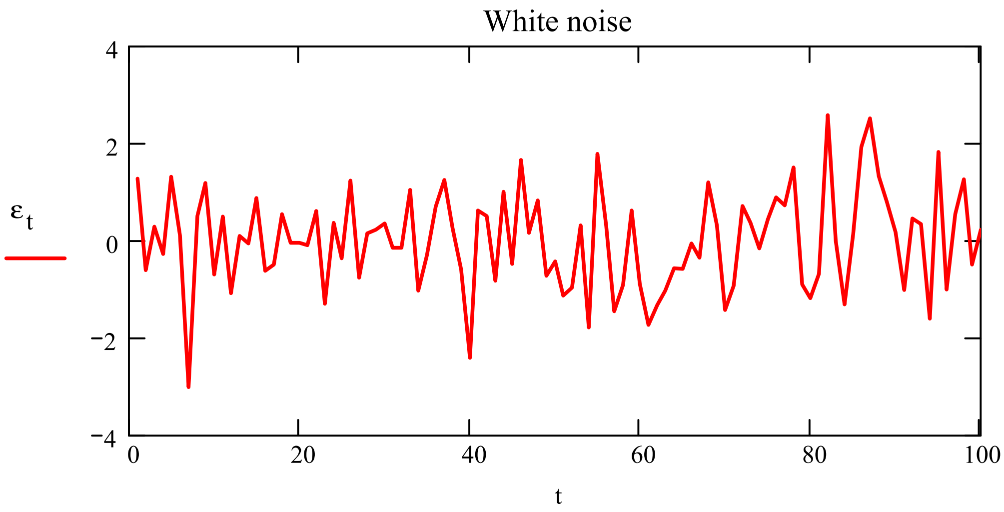
$S_3 \equiv .6$

$S_4 \equiv 0$

Example 3: "White noise"

The random variable ε follows a (0,1)-Gaussian distribution. Now we generate a random sample for $t := 1 .. T_{max}$ with $T_{max} \equiv 100$:

$$\varepsilon_t := \sqrt{-2 \cdot \ln(rnd(1))} \cdot \cos(2 \cdot \pi \cdot rnd(1)) \qquad \Leftarrow \text{ Box-Muller-Transformation}$$



With $w_t := \varepsilon_t$ and $A = 0$, $C = 1$ this time series is a special case of our general linear model. We use this random sequence in the examples 4 - 8 below.

Example 4: Stochastic seasonals

Model: $y_t = \alpha \cdot y_{t-4} + w_t$

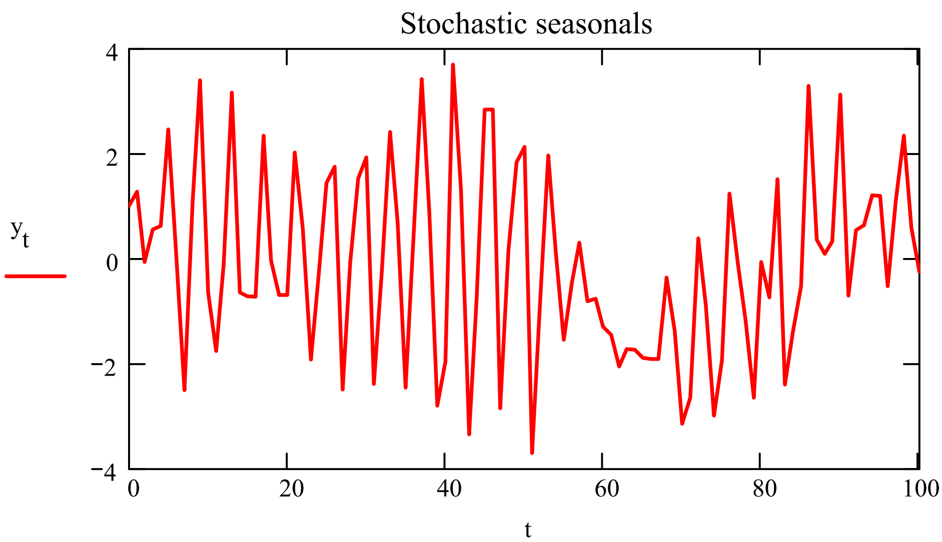
$$\Rightarrow \qquad A := \begin{bmatrix} 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad C := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We use the seasonal pattern of *example 2* together with the random sequence of *example 3*.

$$t := 1..T_{max} \qquad x^{\langle t \rangle} := A \cdot x^{\langle t-1 \rangle} + C \cdot w_t$$

$$t := 0..T_{max} \qquad y_t := \left(x^{\langle t \rangle}\right)_0$$

Notice: If $\alpha = 1$ ($\Leftrightarrow$ unit root) the system tends to display explosive oscillations. If $0<\alpha<1$ ($\Leftrightarrow$ no unit root) the explosive oscillations are no longer present!



$\alpha \equiv .9$

Example 5: Random walk

We call

$$y_t = \sum_{j=0}^{t} w_j$$

a random walk (or martingale process) if w is a white noise variable. If the random walk includes a deterministic trend it is called a **random walk with drift:**

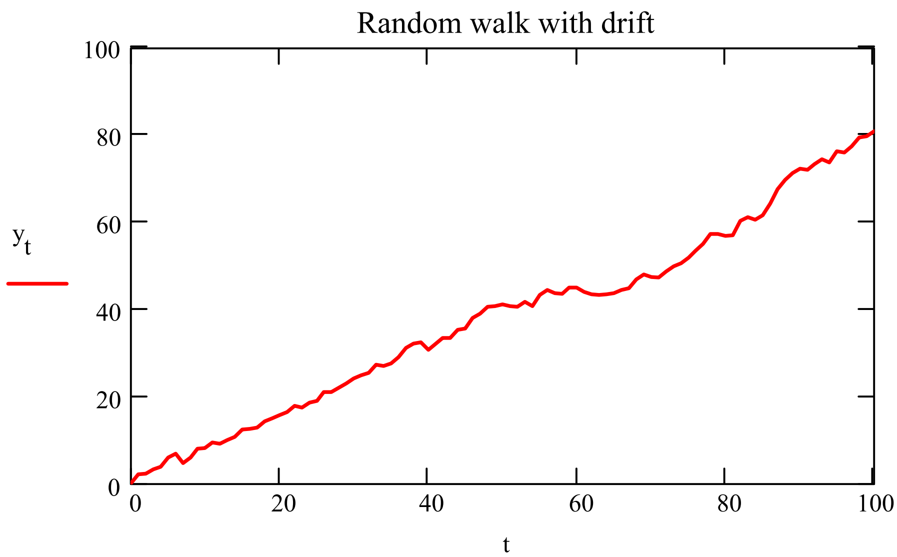
$$y_t = \sum_{j=0}^{t} w_j + \delta_1 + \delta_2 \cdot t$$

Now we redefine this process as a special case of our general model:

$$A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad C := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad x^{<0>} := \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

$$t := 1 .. T_{max} \qquad x^{<t>} := A \cdot x^{<t-1>} + C \cdot w_t$$

$$t := 0 .. T_{max} \qquad y_t := \left(x^{<t>} \right)_0$$



$\delta_1 \equiv 0$

$\delta_2 \equiv .8$

Example 6: Univariate autoregressive process

To represent the AR(4)-model

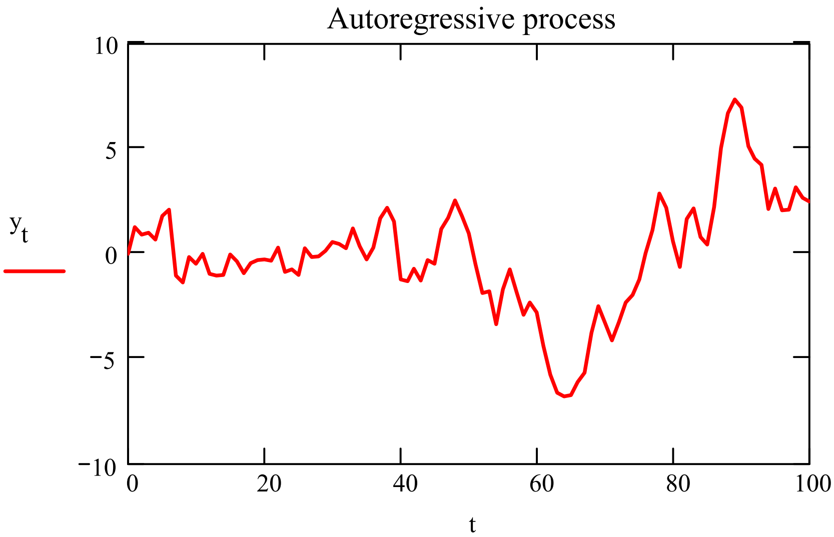
$$y_t = \rho_1 \cdot y_{t-1} + \rho_2 \cdot y_{t-2} + \rho_3 \cdot y_{t-3} + \rho_4 \cdot y_{t-4} + w_t$$

we set:

$$A := \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad C := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad x^{\langle 0 \rangle} := \begin{bmatrix} initial_1 & 0 & 0 & 0 \end{bmatrix}^T$$

$t := 1 .. T_{max}$ $\qquad x^{\langle t \rangle} := A \cdot x^{\langle t-1 \rangle} + C \cdot w_t$

$t := 0 .. T_{max}$ $\qquad y_t := \left(x^{\langle t \rangle} \right)_0$



$initial_1 \equiv 0$

$\rho_1 \equiv 1.2$

$\rho_2 \equiv -.3$

$\rho_3 \equiv 0$

$\rho_4 \equiv 0$

Example 7: Growth with homoskedastic and heteroskedastic noise

The first order autoregression [AR(1)-process]

$$y1_t = \rho \cdot y_{t-1} + w_t$$

with $\rho > 1$ describes a process, where the mean level is growing exponentially at rate ρ per period. The tendency for the randomness dies out, in the sense that the one-step ahead prediction error variance remains unity (⇔ homoskedastic noise).
Now we modify this process a little bit:

$$y2_t = \rho \cdot y_{t-1} + \omega_t \qquad \text{with} \qquad \omega_t := \rho^{t \cdot .5} \cdot w_t$$

This specification makes the variance of ω_t equal to ρ^t. Hence this variance is time dependent (⇔ heteroskedastic noise). Both processes have the transition matrices.

$$A := \begin{bmatrix} \rho & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad C := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For a better comparison, both processes should start with the same initial value:

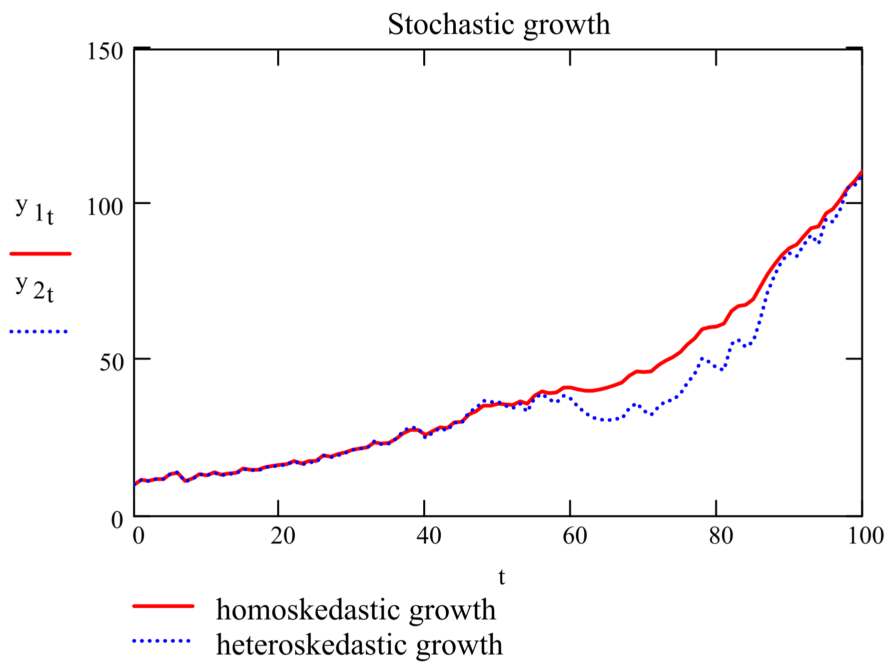
$$x1^{\langle 0 \rangle} := \begin{bmatrix} initial_2 & 0 & 0 & 0 \end{bmatrix}^T \qquad x2^{\langle 0 \rangle} := \begin{bmatrix} initial_2 & 0 & 0 & 0 \end{bmatrix}^T$$

$$t := 1 .. T_{max} \qquad x1^{\langle t \rangle} := A \cdot x1^{\langle t-1 \rangle} + C \cdot w_t$$

$$x2^{\langle t \rangle} := A \cdot x2^{\langle t-1 \rangle} + C \cdot \omega_t$$

$$t := 0 .. T_{max} \qquad y1_t := \left(x1^{\langle t \rangle}\right)_0$$

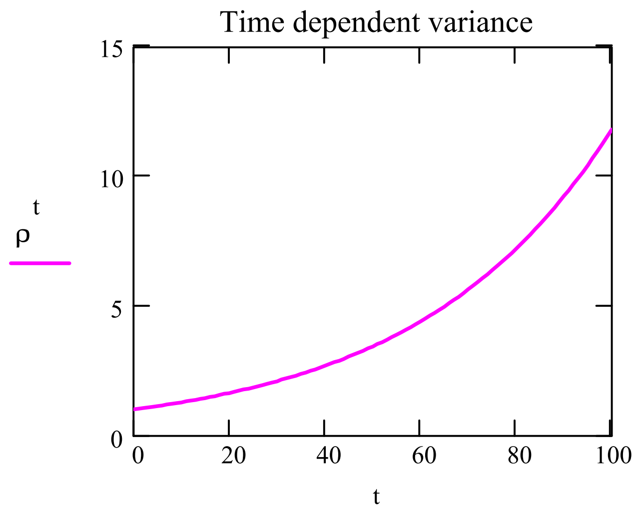
$$y2_t := \left(x2^{\langle t \rangle}\right)_0$$



$initial_2 \equiv 10$

$\rho \equiv 1.025$

Look how the variance $Var(\omega_t) = \rho^t$ increases through time:



Example 8: Univariate autoregressive moving average process

Consider the model of an ARMA(1,1)-process:

$$y_t = \zeta \cdot y_{t-1} + \xi_1 \cdot w_t + \xi_2 \cdot w_{t-1} \qquad \text{with} \quad |\zeta| < 1 \quad \text{and} \quad \left|\frac{\xi_2}{\xi_1}\right| < 1$$

We define the state as:

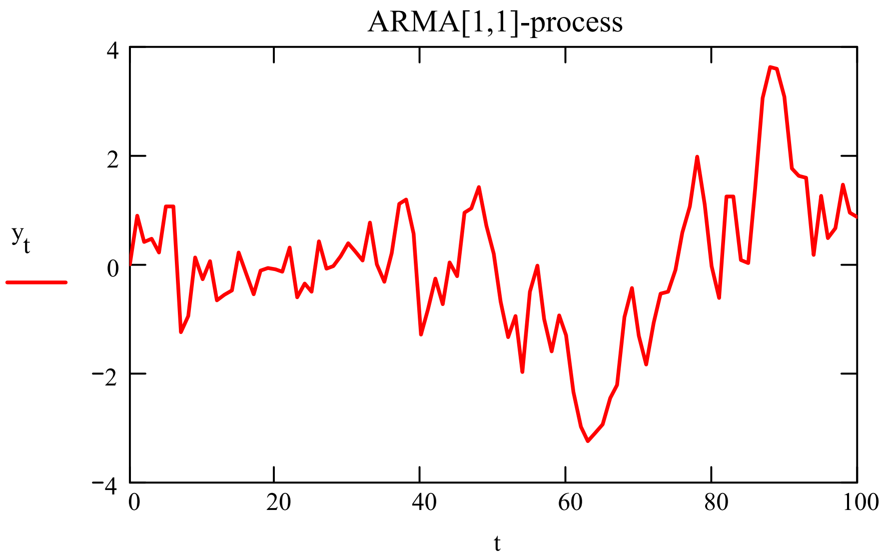
$$x^{<t>} = \begin{bmatrix} y_t \\ \xi_2 \cdot w_t \end{bmatrix}$$

The transition matrices and initial vector are:

$$A := \begin{bmatrix} \zeta & 1 \\ 0 & 0 \end{bmatrix} \qquad C := \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad x^{<0>} := \begin{bmatrix} initial_3 \\ 0 \end{bmatrix}$$

$$t := 1 .. T_{max} \qquad x^{<t>} := A \cdot x^{<t-1>} + C \cdot w_t$$

$$t := 0 .. T_{max} \qquad y_t := \left(x^{<t>}\right)_0$$



$initial_3 \equiv 0$

$\zeta \equiv .8$

$\xi_1 \equiv .7$

$\xi_2 \equiv .1$

Example 9: Vector autoregressive process

We want to simulate a VAR(2)-process:

$$z_{1_t} = a_{11} \cdot z_{1_{t-1}} + a_{12} \cdot z_{1_{t-2}} + b_{11} \cdot z_{2_{t-1}} + b_{12} \cdot z_{2_{t-2}} + w_{1_t}$$

$$z_{2_t} = a_{21} \cdot z_{1_{t-1}} + a_{22} \cdot z_{1_{t-2}} + b_{21} \cdot z_{2_{t-1}} + b_{22} \cdot z_{2_{t-2}} + w_{2_t}$$

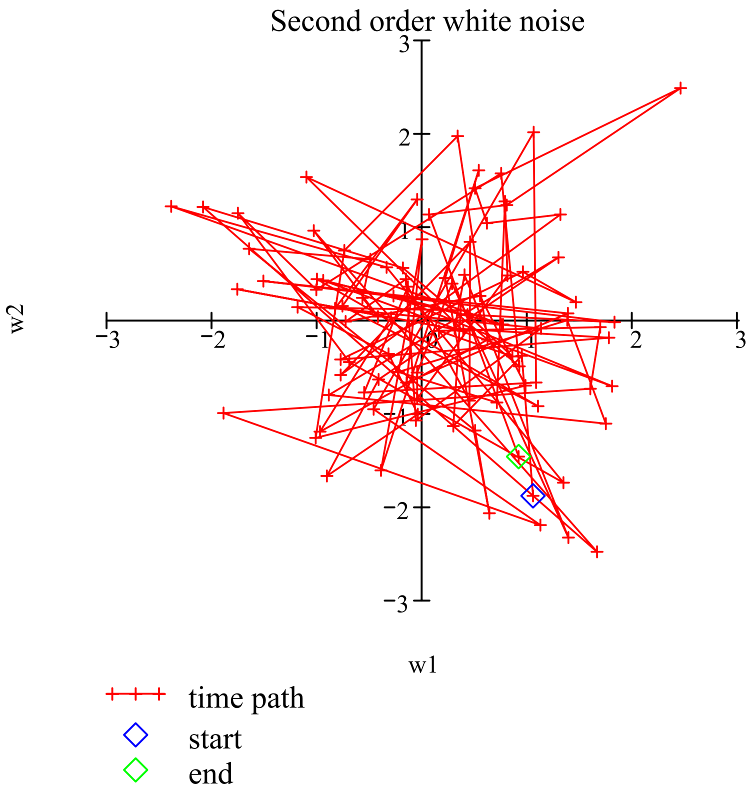
At first we generate $\left(w_{1_t}, w_{2_t}\right)^T$ which is a Gaussian distributed vector white noise with identity covariance matrix:

$t := 1 .. T_{max}$

$w_{1_t} := \sqrt{-2 \cdot \ln(\text{rnd}(1))} \cdot \cos(2 \cdot \pi \cdot \text{rnd}(1))$

⇐ Box-Muller-Transformation

$w_{2_t} := \sqrt{-2 \cdot \ln(\text{rnd}(1))} \cdot \cos(2 \cdot \pi \cdot \text{rnd}(1))$



State vector :

$$x^{\langle t \rangle} = \begin{bmatrix} z1_t \\ z1_{t-1} \\ z2_t \\ z2_{t-1} \end{bmatrix}$$

White noise vector:

$$w^{\langle t \rangle} := \begin{bmatrix} w1_t \\ w2_t \end{bmatrix}$$

Enter parameters of the transition matrix:

$$A := \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

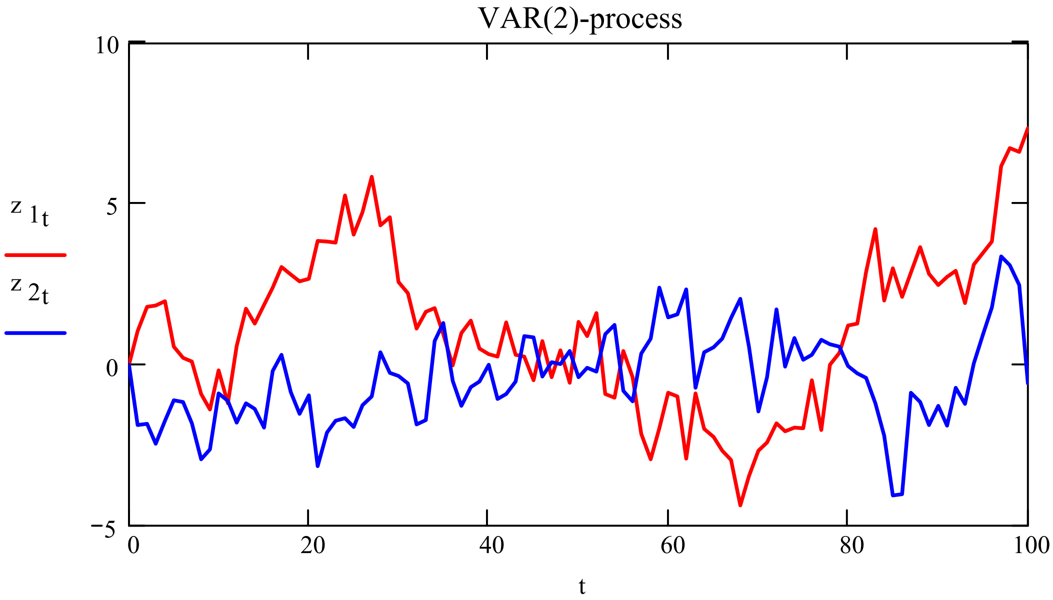
$a_{11} \equiv 0.9$ $a_{12} \equiv 0.05$ $b_{11} \equiv 0.05$ $b_{12} \equiv .01$

$a_{21} \equiv -0.04$ $a_{22} \equiv -0.06$ $b_{21} \equiv 0.75$ $b_{22} \equiv -0.1$

$$C := \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad x^{\langle 0 \rangle} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$t := 1 .. T_{max} \qquad x^{\langle t \rangle} := A \cdot x^{\langle t-1 \rangle} + C \cdot w^{\langle t \rangle}$$

$$t := 0 .. T_{max} \qquad z1_t := \left(x^{\langle t \rangle}\right)_0 \qquad z2_t := \left(x^{\langle t \rangle}\right)_2$$

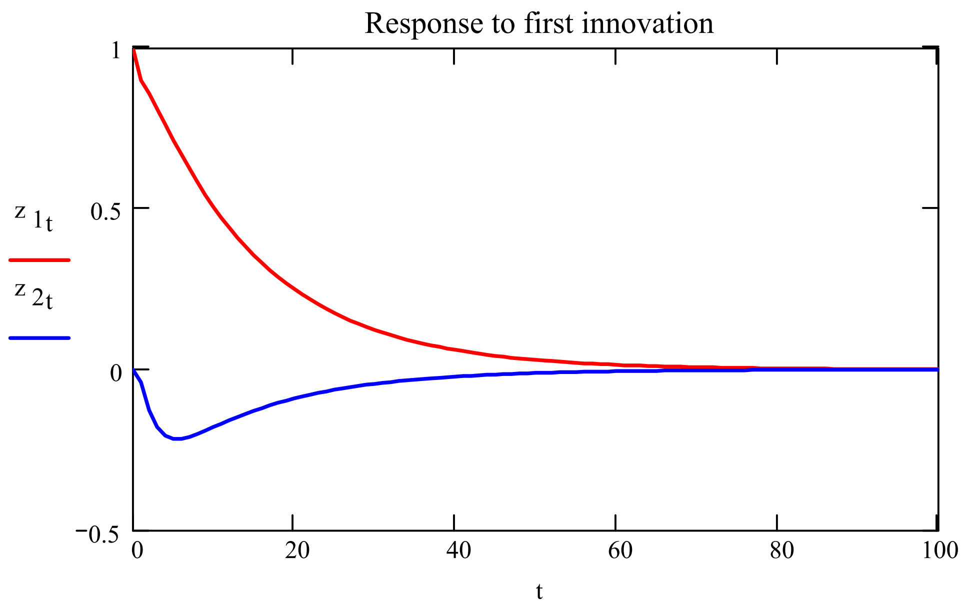


An **impulse response function** depicts the response of the current and future values of z to an imposition of a random shock w (= "innovation"). Here we simulate the response over T_{max} periods of the two variables z_{1_t} and z_{2_t} to the first innovation w_{1_t} .

$$x^{\langle 0 \rangle} := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad w^{\langle t \rangle} := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$t := 1 .. T_{max} \qquad x^{\langle t \rangle} := A \cdot x^{\langle t-1 \rangle} + C \cdot w^{\langle t \rangle}$$

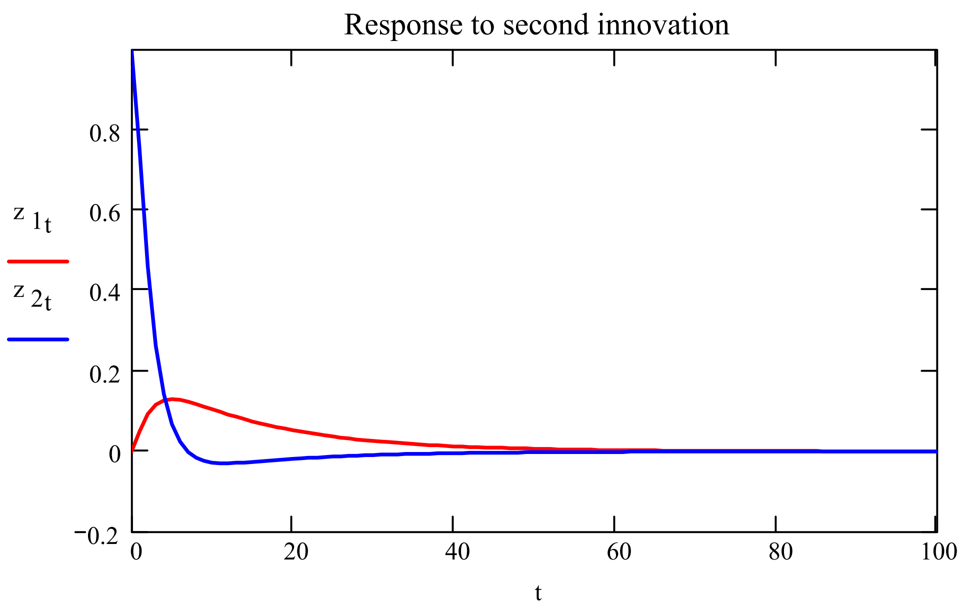
$$t := 0 .. T_{max} \qquad z_{1_t} := \left(x^{\langle t \rangle}\right)_0 \qquad z_{2_t} := \left(x^{\langle t \rangle}\right)_2$$



And now the response to the second innovation w_{2_t} :

$$x^{<0>} := \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad t := 1..T_{max} \qquad x^{<t>} := A \cdot x^{<t-1>} + C \cdot w^{<t>}$$

$$t := 0..T_{max} \qquad z_{1_t} := \left(x^{<t>}\right)_0 \qquad z_{2_t} := \left(x^{<t>}\right)_2$$



It's Your Turn!

Hint: *To get a new random sample of white noise for the same set of parameters choose "compute worksheet" from the MATHCAD menu.*

1. Simulate pure random walks ($\delta_1 = \delta_2 = 0$).
2. Simulate a MA(2)-process ($\zeta = 0$).

Literature:

Hansen, L.P./Sargent, T.J.: Recursive linear models of dynamic economies. Ch. 2., 1998
http://www.stanford.edu/~sargent/hansen.html