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Evolutionary Games

Part I: Symmetric 2×2-Games within a Single Population

Summary:

The analyzed interactions are modeled as pairwise random matchings between individuals in a single polymorphic population. We apply replicator dynamics to the special case of generic symmetric two player games with only two pure strategies. A simple rule then identifies evolutionarily stable strategies (ESS).

Symmetric 2×2-Games

We consider symmetric two-player games. This means

- that there are precisely two player positions (player A and player B),
- that each position has two pure strategies (0 and 1),
- and that the payoff to any strategy is independent of which player position it is applied to.

u_A means the payoff matrix of player A, where the element $u_{A_{i,j}}$ defines the payoff from player A , if he takes his (pure) strategy i and player B takes her (pure) strategy j.

For example:

		Strategy of Player B	
		0	**1**
Strategy of Player A	**0**	$u_{A_{0,0}} \equiv 4$	$u_{A_{0,1}} \equiv 0$
	1	$u_{A_{1,0}} \equiv 5$	$u_{A_{1,1}} \equiv 3$

$$\Rightarrow \qquad u_A = \begin{bmatrix} 4 & 0 \\ 5 & 3 \end{bmatrix}$$

The symmetry requirement from above is equivalent to $u_B := u_A^T$, hence:

$$u_B = \begin{bmatrix} 4 & 5 \\ 0 & 3 \end{bmatrix}$$

Suppose now, that individuals are drawn at random from a large population to play pairwise this symmetric two-person game. Let x be the population share "programmed" to the pure strategy 0. Then 1-x means the population share playing the pure strategy 1. Indeed it is formally immaterial whether an individual interacts with another individual drawn at random from such a polymorphic population, or an individual plays the mixed strategy x. Call $U_i(x)$ the **expected payoff** of an individual playing strategy i (= i-player) given x, then:

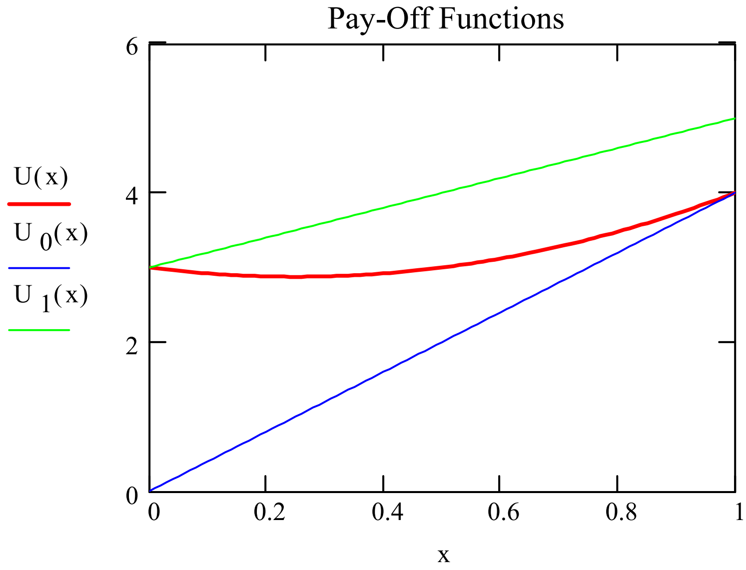
$$U_0(x) := x \cdot u_{A_{0,0}} + (1-x) \cdot u_{A_{0,1}} \qquad \text{and} \qquad U_1(x) := x \cdot u_{A_{1,0}} + (1-x) \cdot u_{A_{1,1}}$$

The associated **population average payoff** (= the payoff of an individual drawn at random from the population) is:

$$U(x) := x \cdot U_0(x) + (1 - x) \cdot U_1(x)$$

In the case of our numerical example we sketch these expected payoffs as functions of x:

$x := 0, .01 .. 1$



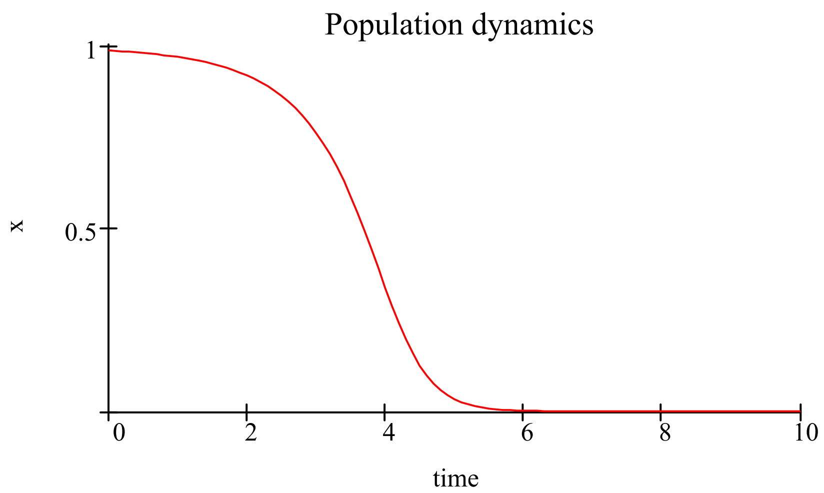
The Replicator Dynamics

Now we assume that this game is continuously repeated over time. In every point of time we interpret x as a population state. Suppose that payoffs represent the incremental effect from playing the game in question on an individual fitness, measured as the number of offspring per time unit. Further the growth rate x'/x of the population share x using strategy 0 equals the difference between the strategy's current payoff and current average payoff in the whole population. Hence we get the replicator dynamics:

$$x' = \frac{d}{d\,time} x = x \cdot \left(U_0(x) - U(x)\right)$$

Enter an initial value of x $x_0 := .99$

and the length of time period to compute the evolutionary process of the population share: $T_{max} := 10$



Categories of Symmetric 2×2-Games

We call a game **doubly symmetric** if u_A**=**u_B. Subtracting $u_{A_{1,0}}$ from the first column and $u_{A_{0,1}}$ from the second column, we **normalize** u_A to obtain a doubly symmetric game:

$$a_1 := u_{A_{0,0}} - u_{A_{1,0}}$$

$$a_2 := u_{A_{1,1}} - u_{A_{0,1}}$$

$$u'_A := \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \qquad u'_B := u'_A{}^T$$

For our numerical example this means:

$$u'_A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \quad \Leftrightarrow \quad u'_B = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

Hence any symmetric 2×2-game can be identified with a point (a_1, a_2) in the R^2-plane.

Following Weibull (1995, p. 40), there are four categories of generic games [by "generic" we mean games in which no payoffs are identical (i.e. $a_1 = 0$ or $a_2 = 0$)]:

Category I: $a_1 < 0$ and $a_2 > 0$

Strategy 1 strictly dominates strategy 0 in any game of this category. Hence all such games are strictly dominance solvable. Strategy 1 is an ESS (**E***volutionarily* **S***table* **S***trategy*).

Category II: $a_1 > 0$ and $a_2 > 0$

All games in this category have three symmetric Nash equilibria. Each of the two pure strategies are ESS. The mixed strategy

$$X_{Nash} := \frac{a_2}{a_2 + a_1}$$

is also a Nash equilibrium but not an ESS.

Category III: $a_1 < 0$ and $a_2 < 0$

Such a game has two strict asymmetric Nash equilibria and one symmetric Nash equilibrium in X_{Nash} but only the last is ESS.

Category IV: $a_1 > 0$ and $a_2 < 0$

This is the reverse case of category I with strategy 0 as an ESS.

Let's check the classification of the game in the numerical example from above:

$$\text{category} := \left| \begin{array}{l} \text{"I"} \quad \text{if } (a_1 < 0)\cdot(a_2 > 0) \\ \text{"II"} \quad \text{if } (a_1 > 0)\cdot(a_2 > 0) \\ \text{"III"} \quad \text{if } (a_1 < 0)\cdot(a_2 < 0) \\ \text{"IV"} \quad \text{if } (a_1 > 0)\cdot(a_2 < 0) \\ \text{"non-generic game"} \quad \text{otherwise} \end{array} \right. \qquad \Rightarrow \qquad \text{category} = \text{"I"}$$

$$X_{Nash} := \left| \begin{array}{l} X_{Nash} \quad \text{if } a_1 \cdot a_2 > 0 \\ \text{"doesn't exist"} \quad \text{otherwise} \end{array} \right. \qquad \Rightarrow \qquad X_{Nash} = \text{"doesn't exist"}$$

It's Your Turn:

Try different payoff matrices, and change the initial population state x_0 to verify the equilibria. For example:

Prisoner's Dilemma:

$$u_A := \begin{bmatrix} 4 & 0 \\ 5 & 3 \end{bmatrix}$$

Coordination Game:

$$u_A := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Hawk-Dove-Game:
(Weibull (1995), example 1.11, p. 27)

Each player has two pure strategies "fight" (= 0) or "yield" (= 1). "Fight" obtains payoff $v > 0$ when played against "yield", which in this case obtains payoff 0. Each player has an equal chance of winning a fight, and the cost of losing a fight is $c > 0$. When played against itself, strategy 0 thus gives payoff v with probability 0.5 and payoff -c with probability 0.5. Hence the expected payoff of strategy 0 against itself is (v - c)/2. When both players yield, each gets payoff v/2. The resulting payoff matrix is thus:

$$u_A := \begin{bmatrix} \frac{v-c}{2} & v \\ 0 & \frac{v}{2} \end{bmatrix}$$

where for instance:

$v \equiv 4$ $\quad$ $c \equiv 6$ $\quad \Rightarrow \quad$ $u_A = \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix}$

Literature:

Weibull, J.W.: Evolutionary Game Theory. Cambridge (MA)/London, 1995.