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# Evolutionary Games

## Part III: Tit for Tat


### Summary:

As is well known, in the "prisoner's dilemma paradox" each player has a dominant strategy (called "defection") which leads to an inefficient outcome, because if both players were to choose "cooperation", both were be better off than by jointly playing their dominant strategy. But if this game is repeated within an infinite (or even unknown) time horizon, and if players don't discount future pay-offs too much, a wide range of new possibilites arises strengthening cooperative behaviour. The computational experiments of Axelrod (1984) demonstrated that the simple strategy of "tit for tat" suggested by Anatol Rapoport fared relatively better in the long run than more sophisticated strategies. We shall focus on an example from Vogan-Redondo (1996) that illustrates the operation of evolutionary forces in the rise of cooperative behaviour. This model underlines an important point, which is encountered in the work of Vogan-Redondo: "the (natural) introduction of noise into an evolutionary model often brings 'order rather than chaos' into the model's behaviour."

## Repeated Prisoner's Dilemma (RPD)

Individuals from a single large population are randomly matched every period to play an RPD with an infinite time horizon. They adopt one of the following **strategies**:

C = choose cooperation irrespective of past history
D = choose defection irrespective of past history
T = choose "tit for tat", which starts by cooperating and then responds by matching the opponent's action in the preceding stage

The intertemporal pay-off of each player equals the dicounted sum of their correponding stream of pay-offs with the discount rate $\delta$. Then the discounted pay-offs are multiplied by the factor $(1-\delta)$, such that they remain within the convex hull of the stage pay-offs:

**Pay-off matrix:**

| | C | D | T |
|---|---|---|---|
| C | 3 , 3 | 0 , 4 | 3 , 3 |
| D | 4 , 0 | 1 , 1 | $4 - 3\cdot\delta , \delta$ |
| T | 3 , 3 | $\delta , 4 - 3\cdot\delta$ | 3 , 3 |

Enter **discount rate:** $\delta := \frac{2}{3}$

Further, we define:

x = frequency of D-strategists
y = frequency of T-strategists
z = 1 - x -y = frequency of C-strategists

with $0 \le x , y \le 1$

(Because x+y+z = 1, the values of x and y give a sufficient description of the state of the system.)

Expected pay-off for a C-strategist:

$$\Pi_C(x,y) := (3\cdot(1-x-y) + 0\cdot x + 3\cdot y) \rightarrow 3 - 3\cdot x$$

Expected pay-off for a D-strategist:

$$\Pi_D(x,y) := (4\cdot(1-x-y) + 1\cdot x + (4 - 3\cdot\delta)\cdot y) \rightarrow 4 - 3\cdot x - 2\cdot y$$

Expected pay-off for a T-strategist:

$$\Pi_T(x,y) := (3\cdot(1-x-y) + \delta\cdot x + 3\cdot y) \rightarrow 3 - \frac{7}{3}\cdot x$$

Mean expected pay-off across all strategies:

$$\Pi_{mean}(x,y) := \left[\begin{array}{l}(1-x-y)\cdot\Pi_C(x,y) \ldots \\ +x\cdot\Pi_D(x,y) + y\cdot\Pi_T(x,y)\end{array}\right] \text{vereinfachen} \rightarrow 3 - 2\cdot x - \frac{4}{3}\cdot y\cdot x$$

## Replicator dynamics

An individual-based kind of mutation is assumed, that is statistically independent across individuals and time: Each one of the three strategies is introduced in the population irrespective of any pay-off considerations by a mutation rate $\theta > 0$ with the same a priori probability 1/3. Otherwise with the complementary probability 1-$\theta$ each individual "stays alive" producing offspring in proportion to its respective pay-offs. With $\theta = 0$ the game is unperturbed (= no mutations arise.)

Replicator dynamics for D-strategists:

$$\frac{dx}{dtime} = x' = (1-\theta)\cdot\left(x\cdot\Delta_D(x,y)\right) + \theta\cdot\left(\frac{1}{3} - x\right)$$

with

$$\Delta_D(x,y) := \Pi_D(x,y) - \Pi_{mean}(x,y) \text{ vereinfachen } \rightarrow 1 - x - 2\cdot y + \frac{4}{3}\cdot y\cdot x$$

Replicator dynamics for T-strategists

$$\frac{dy}{dtime} = y' = (1-\theta)\cdot\left(y\cdot\Delta_T(x,y)\right) + \theta\cdot\left(\frac{1}{3} - y\right)$$

with

$$\Delta_T(x,y) := \Pi_T(x,y) - \Pi_{mean}(x,y) \text{ vereinfachen } \rightarrow \frac{-1}{3}\cdot x + \frac{4}{3}\cdot y\cdot x$$

## Stationary Equilibria

To determine the stationary points of the replicator dynamics the condition x'= y'= 0 must hold.

The function $y = \phi_1(x,\theta)$ stands for all x and y with x' = 0:

$$\phi_1(x,\theta) := \left[(1-\theta)\cdot\left(x\cdot\Delta_D(x,y)\right) \ldots + \theta\cdot\left(\frac{1}{3} - x\right)\right] \text{ auflösen, } y \rightarrow \frac{1}{2}\cdot\frac{\left(3\cdot x - 3\cdot x^2 - 6\cdot\theta\cdot x + 3\cdot\theta\cdot x^2 + \theta\right)}{((-1+\theta)\cdot x\cdot(-3+2\cdot x))}$$

The function $x = \phi_2(y, \theta)$ stands for all x and y with y' = 0:

$$\phi_2(y,\theta) := \begin{bmatrix} (1-\theta)\cdot\left(y\cdot\Delta_T(x,y)\right) \ldots \\ +\theta\cdot\left(\frac{1}{3} - y\right) \end{bmatrix} \text{auflösen}, x \rightarrow -\theta\cdot\frac{(-1+3\cdot y)}{((-1+\theta)\cdot y\cdot(-1+4\cdot y))}$$

Thus, the intersection points of this functions are stationary equilibria!

If **θ = 0 (unperturbed model)** there exists a continuum of such stationary points: All points in the set {(0,y) : 0≤y≤1} are stationary. But only the subset with y≥Y where

$$Y := \left(\Pi_D(0,y) = \Pi_T(0,y)\right) \text{auflösen}, y \rightarrow \frac{1}{2}$$

satisfies Lyapunov stability. There are also two isolated stationary points. One of them is (X,1-X), where

$$X := \left(\Pi_D(x,1-x) = \Pi_T(x,1-x)\right) \text{auflösen}, x \rightarrow \frac{3}{4}$$

This point is not even Lyapunov stabel. The only asymptotically stable equilibrium is the single point (0,1).

In the case of **"noisy dynamics" (θ > 0)** only an exact frequency with y>Y of T-strategists will support a (in fact asymptotically) stable cooperative outcome. Therefore, also small perturbations caused by mutation may have a significiant effect on evolutionary dynamics!

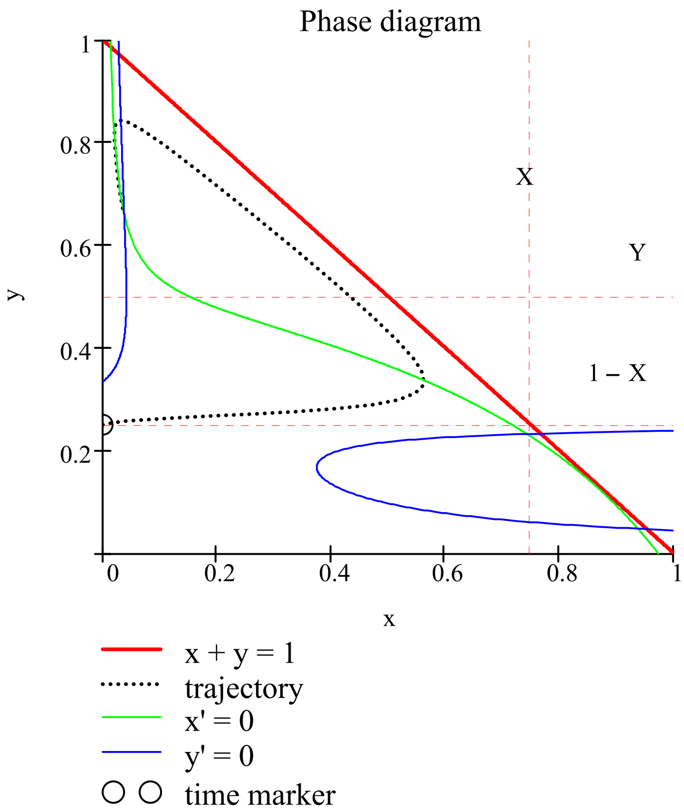
## Simulation

Now we are able to examine the global evolutionary dynamics with the help of a phase diagram. The triangle formed by the points (0,0), (1,0) and (0,1) represents the state space. The green curve of the isocline $\phi_1$ and the blue one of $\phi_1$ show where the stability conditions hold. Intersection points of both lines are stationary points. Starting from an initial point the black dotted line marks the evolutionary path of x and y.

Choose initial values of x and y below to sketch the evolutionary pathes for $T_{max} \equiv 80$ time periods. The maximum number of values plotted are $n_{max} \equiv T_{max}$. Animate the figure to see the direction and the relative speed of the process. The red point will run around the orbit, showing the position of employment rate and workers' share at each time.

WRITE("t.tmp") := -1 ⇐ To set back the animation to time = 0, click with your mouse on the red field and press the F9-key.

Animation := READ("t.tmp") ⇐ To animate the figure below, click with your mouse on the yellow field and hold down the F9-key.




**Mutation rate:**

$\theta \equiv .04$

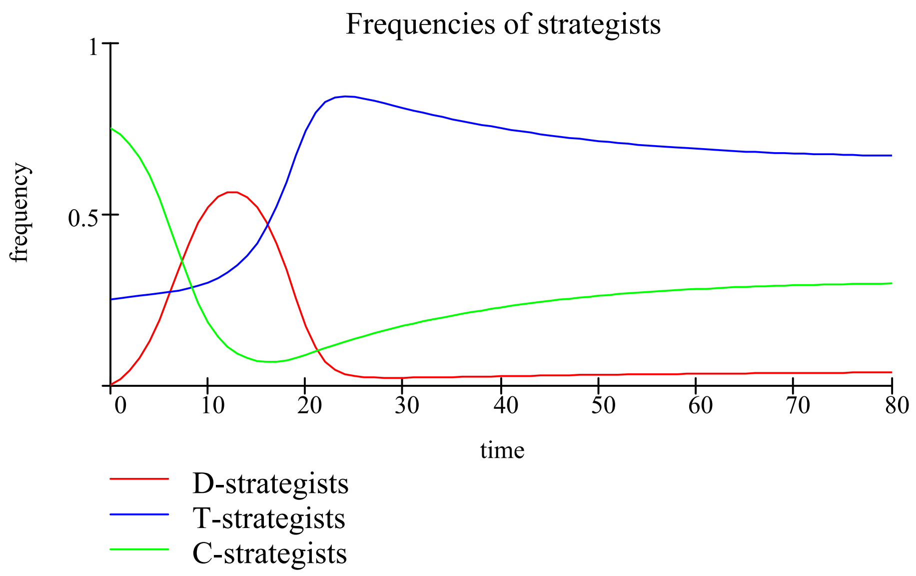
**Initial values:**

$x_{init} \equiv 0$

$y_{init} \equiv .25$

Actual period:

time = 0




## It's Your Turn!

Try different initial values to analyze the global dynamics of the model.

Decrease the mutation rate, till θ = 0.

Decrease the discount rate.

## Literature:

Axelrod, R.: The Evolution of Cooperation.  New York 1984.

Vega-Redondo, F.: Evolution, Games, and Economic Behaviour. Oxford University Press 1996, 72 -79.